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Recursion Theory and Computational Complexity

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C O N T E N T S

CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C.I.M.E.)

ADMISSIBLE RECURSION THEORY

STEVE HOMER

ADMISSIBLE RECURSION THEORY

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INTRODUCTION

The purpose of these lectures is to develop some deeper results in α-recursion theory which will hold in somewhat more general setting than $L(\alpha)$ and in particular in many other admissible sets and structures. In addition, I will briefly mention some applications to structures which arise from other areas of recursion theory and which are inadmissible.

An important underlying idea behind many of these theorems is the notion of a dynamic argument. In general what is meant by this is the following: In ω-recursion theory, the starting point for all of these generalizations, the exceedingly strong closure properties of ω are used in almost every construction, often without a second glance. Almost any operation on finite sets yields a finite set and in particular the image of a finite set under any function is finite. When we consider α-recursion theory, α Σ_1-admissible, we of course loose a great deal of the closure present in ω-recursion theory. However we still have a certain weak closure property - the image of an α-finite set under a $\Sigma_1(L_\alpha)$ function is α-finite. But as soon as we do a construction in which a $\Sigma_2(L_\alpha)$ (or $\Sigma_3(L_\alpha)$....) function arises, and such does occur in almost every priority argument, then we immediately run into trouble. (We will see a concrete example of this shortly when we start talking about Post's problem in this setting.)

While we have lost the strong closure properties in $L(\alpha)$ we have gained the use of much work in set theory, mainly by Gödel and Jensen, in which many deep properties of L have been developed. It is just these properties of L which often save us when we run into trouble because of a lack of closure under certain functions.

Now some properties of L work, under suitable conditions, in other settings, while others are really peculiar to L. In particular many of the deeper results about L depend on taking Skolem hulls of certain sets in $L(\alpha)$, taking the tranitive collapse of that Skolem hull, and being able to determine exactly what that transitive collapse looks like, namely, an initial segment of $L(\alpha)$ (an $L(\gamma)$ for some $\gamma \leq \alpha$).

These collapsing arguments tend to be very specific to L and almost never work in other settings. In particular, they fail when the universe is changed by constructing L relative to a given predicate. So, to get constructions to work in more general settings, we need to eliminate, if at all possible, these collapsing arguments peculiar to L. We want to give a "dynamic" argument (basically more similar to the original one for ω), which views r.e. sets is being listed and increasing in an effective manner, not as being defined by a Σ_1-formula which is really the crucial property making collapsing arguments work.

In order to make this more explicit I want now to turn to a couple of concrete examples. Both examples depend upon the same method and so the second will be given in much less detail. After presenting these examples I will indicate some extensions and applications of these results to other areas and in particular to admissible sets and structures. We will also, and of course this is the reason these methods were first developed, gain some knowledge of the structure of α-r.e. degrees.

In what follows I am assuming a familiarity with the basic facts and definitions of α-recursion theory. These are given in the first of Sacks' lectures in this volume. For a more detailed account see the papers by Simpson [18] or Shore [14].

1. Post's Problem for α-Recursion Theory

Let α be a Σ_1-admissible ordinal. We want to prove the following theorem.

Theorem: There exists two α-r.e. sets which are incomparable with respect to $\leq_\alpha$. This theorem was originally proved by Sacks and Simpson [11]. Their argument depended heavily on using properties of Skolem hulls in L.

I will present another way of proving this theorem based largely on ideas of R. Shore [14], [16]. Shore's ideas were applied to this problem by Simpson [19]. The proof is more dynamic, more "constructive" if you will, than the original proof and, as we will see, is more adaptable to other settings. It is not so dependent on the special properities of initial segments of L.

The argument will be presented by starting out with the basic ideas for solving Post's problem from ω-recursion theory.

I assume some familiarity with that argument. As we try to carry out the argument in the setting of $L(\alpha)$ we will meet with various difficulties for which we will propose solutions. Finally, we will put all of this together to get the actual construction and proof.

Now, we are going to construct two α-r.e. sets A and B. We require that $A \not\leq_\alpha B$ and $B \not\leq_\alpha A$. In fact we will construct them to satisfy the stronger incomparability, $A \not\leq_{w\alpha} B$ and $B \not\leq_{w\alpha} A$. That is, for each $e \in L(\alpha)$, we want to ensure that

$$S_e^A : \{e\}^A \neq B \qquad \text{and}$$

$$S_e^B : \{e\}^B \neq A.$$

A bit of notation is necessary here. Requirements of the form S_e^A are called <u>A-requirements</u>, requirements of the form S_e^B are <u>B-requirements</u>. In these requirements we are identifying a set C with its characteristic function,

$$C(x) = \begin{cases} 0 & \text{if } x \notin C \\ 1 & \text{if } x \in C \end{cases}$$

For any set $C \subseteq \alpha$, we let $\bar{C} = \alpha - C$. In the construction of A and B we let $A^\sigma (B^\sigma)$ = set of elements enumerated in A (B) by stage σ. Finally let $A^{<\sigma} = \bigcup_{\delta<\sigma} A^\delta$, $B^{<\sigma} = \bigcup_{\delta<\sigma} B^\delta$.

The method used to satisfy the above requirements is the same as for ω-recursion theory. Consider $\{e\}^A \neq B$. At various stages of the construction it will appear that we can use A to to enumerate $\bar{B}$. That is, at a stage σ we will see that $\{e\}^{A^{<\sigma}}(x)=0$ for some x which is in $\overline{B^{<\sigma}}$. We would like to put x into B^σ, insuring that $B(x) = 1$, and at the same time try to keep the computation $\{e\}^{A^{<\sigma}}(x) = 0$ correct. This computations uses

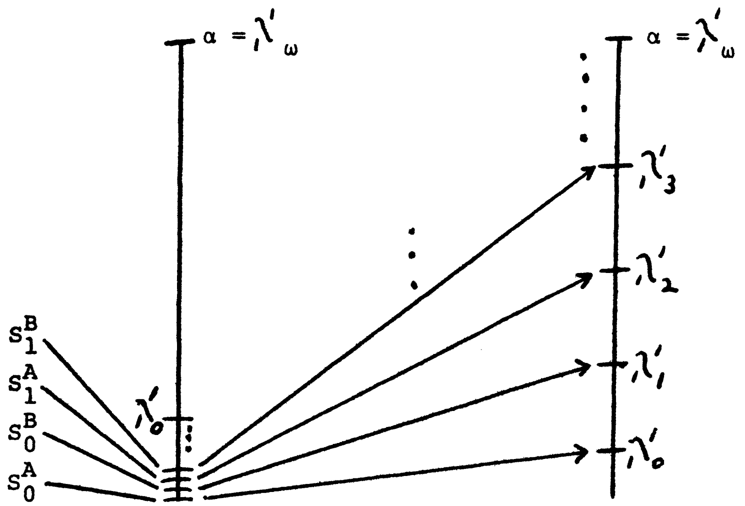
α-finite subsets of $A^{<\sigma}$ and $\overline{A^{<\sigma}}$ and so we try to keep the α-finite subset of $\overline{A^{<\sigma}}$ from intersecting A.

Of course, we do the same for requirements $\{e'\}^B \neq A$. For such requirements we try to keep elements out of B and to put them into A, so they tend to conflict with the A-requirements. Such conflicts arise in classical recursion theory as well and are resolved by giving priority to a requirement S_e^A over a requirement $S_{e'}^B$, if, say, $e \leq e'$. If we do this, an inductive argument then proves that each requirement is eventually acted upon successfully. In addition, for any e, there is a stage of the construction by which any action taken on behalf of requirements $S_{e_0}^A$ or $S_{e_1}^B$, where e_0, $e_1 < e$, has already been taken. In other words all activity on behalf of such requirements has already settled down.

This gives rise to the first major problem in doing this argument for α. The function taking the e^{th} requirement to the first stage at which it has settled down is $\Sigma_2(L(\alpha))$ and so we can't use the admissibility to get a bound on the activity of the requirement.

For example, lets look at the particular admissible ordinal $\alpha = \aleph'_\omega$. It could be that S_0^A does not stop acting until after stage $\aleph'_0$, S_0^B until after stage $\aleph'_1$, the S_1^A until after stage $\aleph'_2$, etc. If this were the case then there might be no stage past which the requirement S_ω^A is never injured, there being no bound on the activity of the first ω-many requirements.

Picture:



The problem is that our priority listing of requirements is too long. If we just list our requirements in the usual, most straightforward way, S_0^A, S_0^B, S_1^A, S_1^B, S_2^A, ..., S_e^A, S_e^B, --, $e<\alpha$, an initial segment of this list may never settle down.*

Shore devised the following method to avoid this problem. Note that requirements of the same kind do not conflict. We take advantage of this trivial fact by arranging our requirements in groups or blocks, each block containing requirements

* In fact it can be shown that this problem is avoided for this particular α. However for many admissible α this is the crucial problem.

of the same type. We essentially treat each block as a single requirement. So we will have blocks R_0^A, R_0^B, R_1^A, R_1^B,... where R_e^A is a block of A-requirements, R_e^B a block of B-requirements. Each block consists of lots of requirements but as they don't conflict we give each requirement the same priority. By doing this we can, for troublesome α, get away with less than α-many priorities and blocks and so will be able to bound the activity of an initial segment of blocks, even though the function giving this bound is $\Sigma_2(L(\alpha))$.

How short do we need to make the list of blocks? Short enough so that no Σ_2 function from an initial segment of the list of blocks to α is unbounded.

Definition: Let $\lambda = \underline{\Sigma_2\text{ cf}(\alpha)}$ = least $\beta \leq \alpha$ such that the there is an $f:\beta \to \alpha$, $f\ \Sigma_2(L(\alpha))$ and range(f) unbounded in α.

We will make the list of blocks have length λ, thus ensuring the above property.

Now this takes care of part of the problem as there are now not too many blocks. Within each block, requirements don't conflict. We need to make sure that no block is too large so that we can find a stage at which all activity in a given block has settled down. Let's look at the very first block. The requirements in this block, since they have the highest priority, are all acted upon once at most. The set of requirements which are acted upon is α-r.e.. If we could ensure that this set is α-finite, we could then conclude there is a stage past which no requirement in the first block is acted upon. How can we do this? Make the block bounded below α^*, for any α-r.e. set bounded below α^* is α-finite.

(For a proof see Devlin [3] or Sacks' lectures in this volume.) To accomplish this we will use a projection $p: \alpha \xrightarrow{1-1} \alpha^*$ to enumerate our requirements in a list of length α^*. Our blocks will then be subsets of α^*.

We have one more problem to surmount. We would like to divide α^* into $\Sigma_2 cf(\alpha)$ many blocks. We could divide α^* into $\Sigma_2 cf(\alpha^*)$ many blocks, where $\Sigma_2 cf(\alpha^*)$ = least $\beta \leq \alpha$ such that there is an $f: \beta \to \alpha^*$, f $\Sigma_2(L(\alpha))$ and range(f) unbounded in α^*, but this could be too many blocks. The following crucial lemma says that this cannot happen.

<u>Lemma</u>: $\Sigma_2 cf(\alpha) = \Sigma_2 cf(\alpha^*)$.

Proof (Simpson [19]):

Let $p: \alpha \xrightarrow{1-1} \alpha^*$ be α-rec. Using the admissibility of α it is straightforward to show that if $X \subseteq \alpha$ is unbounded then $p[X]$ is unbounded in α^*.

Now let $\delta \leq \alpha$, $g: \delta \longrightarrow \alpha$ be $\Sigma_2(L(\alpha))$ with range(g) unbounded in α. The $p \circ g: \delta \longrightarrow \alpha^*$ is $\Sigma_2(L(\alpha))$ and by the above fact, range($p \circ g$) is unbounded in α^*. So $\Sigma_2 cf(\alpha^*) \leq \Sigma_2 cf(\alpha)$.

Conversely let $h: \gamma \longrightarrow \alpha^*$ be $\Sigma_2(L(\alpha))$ and range(h) unbounded in α^*. Define $k: \gamma \longrightarrow \alpha$ by, for $\nu < \gamma$,

$$k(\nu) = \bigcup \{\sigma \mid p(\sigma) < h(\nu)\}.$$

k is $\Sigma_2(L(\alpha))$ and range(k) is unbounded in α.

So $\Sigma_2 cf(\alpha) \leq \Sigma_2 cf(\alpha^*)$. ⊣

2. The Construction and Proof

Using the above lemma we can now put all of our considerations together and give the blocking construction.

Let $H\colon \lambda = \Sigma_2 cf(\alpha) = \Sigma_2 cf(\alpha^*) \longrightarrow \alpha^*$ be a $\Sigma_2(L(\alpha))$ function.

Without loss of generality we can assume,

1. $H(0) \leq H(1) \leq \ldots \leq H(\gamma) \leq \ldots < \alpha^*$ and
2. $H(\delta) = \bigcup_{\gamma<\delta} H(\gamma)$, for any limit ordinal δ.

For details of this and other properties of the blocking function H and its approximation see Simpson [19] .

Using the above H function we can now say precisely what our blocks R_0^A, R_0^B, R_1^A, R_1^B, . . . of requirements will be.

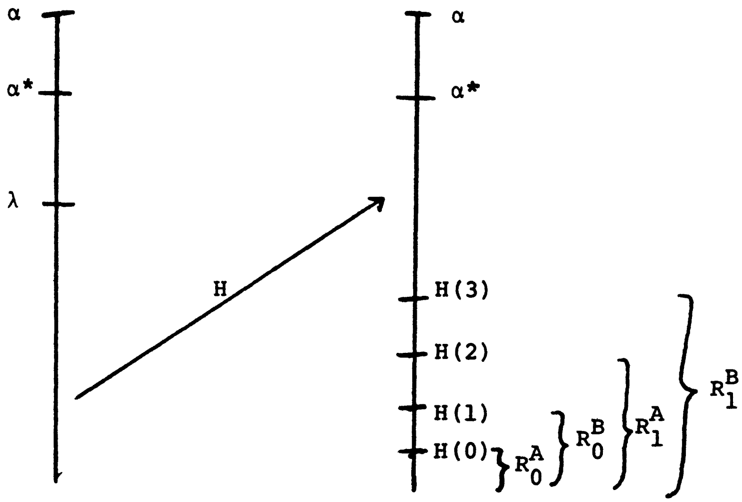
$R_0^A = S_0^A, S_1^A, S_2^A, \ldots, S_e^A, \ldots \qquad e < H(0)$

$R_0^B = S_0^B, S_1^B, \ldots, S_e^B, \ldots \qquad e < H(1)$

$R_1^A = S_0^A, S_1^A, \ldots, S_e^A, \ldots \qquad e < H(2)$

$R_2^B = S_0^B, S_1^B, \ldots, S_e^B, \ldots \qquad e < H(3)$

The picture is:



We will now give the construction of the α-r.e. sets

A and B. In this construction three "details" are left out. These require some care but tend to obscure the proof so I will briefly describe them here and leave them to you.

1. It is convenient to keep the witnesses x for different requirements disjoint and necessary to make sure we have an endless supply of them. To do this, break α up into α-many disjoint α-rec. sets, each unbounded in α. Assign one such set to each requirement in each block. The witnesses for a requirement are constrained to come from the associated set.

2. Since H is $\Sigma_2(L(\alpha))$ we cannot effectively calculate its value. Instead we approximate H with an α-rec. function G. G has the property that when restricted to an initial segment of λ, G equals H on all but initial segment of stages. We will refer to H in the construction but keep in mind that what we really do is calculate values of G, a "nice" approximation to H.

3. Using the α-rec. projection $p\colon \alpha \xrightarrow{1-1} \alpha^*$ we can assume that the requirements are enumerated via ordinals less than α^*. So from now on we assume we have requirements S_e^A, S_e^B where $e < \alpha^*$. Of course many of these requirements are undefined but all of the original requirements are contained in this enumeration.

The construction takes place in α-many stages. Initially $A^0 = B^0 = \emptyset$.

Stage σ:

Step 1: For each odd $\nu < \lambda$ proceed as follows. Find the least $e < H(\nu)$ and the least $x < \sigma$ in the witness set for S_e^A such that

1. There is no requirement for S_e^A.
2. $B^{<\sigma}(x) = 0 = \{e\}_\sigma^{A^{<\sigma}}(x)$.
3. x is not being kept out of B for some requirement in an earlier block. That is, a requirement in some block determined by $H(\nu')$ where $\nu' < \nu$.

If no such e and x exist do nothing and proceed with step 2.

Otherwise put x in B^σ and create a negative requirement keeping out of A all elements of $A^{<\sigma}$ used in the computation of $\{e\}_\sigma^{A^{<\sigma}}(x) = 0$. If x is contained in any negative requirement on B that requirement is destroyed.

Step 2: Do the same as in step 1 only for even $\nu < \lambda$ interchanging the roles of A and B.

The following claim is the main ingredient in showing that the construction works.

Claim 1: For each $\nu < \lambda$ there is a stage σ_0 such that all activity on all blocks $\leq \nu$ has ceased. i.e. At no stage past σ_0 is any requirement in any block $\leq \nu$ injured or acted upon.

Proof: By induction on $\nu < \lambda$.

Case 1: $\nu = 0$

The proof here is the same as in the successor case 2.

Case 2: $\nu = \nu' + 1$

By induction there is a stage σ_1 such that by stage σ_1 all blocks up to and including ν' have settled down. Then past stage σ_1 no requirement in block ν can be injured. Consider the set $I = \{e | e < H(\nu)$ and for some $\sigma_2 \geq \sigma_1$,

S_e^A is acted upon at stage σ_2} (Without loss of generality we are assuming here that ν is odd.) I is α-r.e. and $I \subseteq H(\nu) < \alpha^*$, hence I is α-infinite.

Define $f: I \longrightarrow \alpha$ by

$f(e)$ = least σ (S_e^A is acted upon at stage σ). Then $f[I]$ is bounded by the desired stage σ_0.

Case 3: ν a limit ordinal

Define $g: \nu \longrightarrow \alpha$ by, for $\gamma < \nu$, $g(\gamma)$ = least σ (block γ has settled down by stage σ). Then g is $\Sigma_2(L(\alpha))$ and so since $\nu < \lambda$, $g[\nu]$ is bounded by the desired stage σ_0. ⊣

We can now prove the theorem.

Claim 2: $A \not\leq_{w\alpha} B$ and $B \not\leq_{w\alpha} A$.

Proof: We will show $A \not\leq_{w\alpha} B$.

Assume not, then $A = \{e\}^B$ for some $e < \alpha^*$. Let $\nu < \lambda$ be the least even ordinal such that $e < H(\nu)$. Let σ_0 be a stage past which all blocks $\leq \nu$ have settled down. Then there is an $x < \sigma_0$ in the witness set for S_e^B as a ν-requirement and a stage $\sigma_1 > x$, σ_0 such that $\{e\}_{\sigma_1}^{B^{<\sigma}}(x) = A^{<\sigma}(x) = 0$. As $x > \sigma_0$, x is not in any negative requirement for A of higher priority and $x \notin A$. So at stage σ_1 either x is put into A^{σ_1}, contradicting $x \notin A$ or there is a requirement for S_e^B present at stage σ_1 and hence never injured, insuring that $\{e\}^B \neq A$. ⊣

3. The Splitting Theorem

The blocking method described in the previous sections was first developed by Shore to prove the following generalization of Sacks' splitting theorem.

Theorem (Shore [16]): Let C and D be regular α-r.e. sets with D not α-recursive. Then there exists α-r.e. sets A and B such that $A \cap B = \emptyset$, $A \cup B = C$, $A,B \leq_\alpha C$, $D \not\leq_\alpha A$ and $D \not\leq_\alpha B$.

All of the usual corollaries of the splitting theorem concerning Turing degrees are true for α-degrees as a consequence of this theorem. For instance below any non-zero α-r.e. degree there are two incomparable α-r.e. degrees.

We have already encountered in our proof of Post's problem all of the major obstacles to proving Shore's splitting theorem. This being so I will constrain myself to briefly mentioning the main ideas of the proof.

The requirements which we satisfy are of three types:

S_e^A: $\{e\}^A \neq D$

S_e^B: $\{e\}^B \neq D$

R: $x \in C \to x \in A \cup B$

As usual we construct A and B in α-many stages. To satisfy S_e^A we follow Sacks' original strategy of trying to preserve the equality $\{e\}^A = D$ on longer and longer initial segments of α. The idea here is that if we succeed in preserving equality all the way up to α then we could compute D α-recursively. Since D is not α-recursive this can't happen, but if $\{e\}^A = D$ we could show that the preservation up to α would succeed. So S_e^A must be satisfied. S_e^B is handled similarly. To satisfy R we insist that when an ordinal x appears in the α-recursive enumeration of C at some stage of the construction, it is immediately put into exactly one of A or B. Such an x is put into A or B so as to preserve the highest priority negative requirements on A or B which contains x. These negative requirements are created for A and B as a result of action taken for S_e^A and S_e^B.

The construction is carried out just as in the above proof of Post's problem. As there, the requirements S_e^A and S_e^B are enumerated via ordinals less than α* and α* is divided into $\Sigma_2 cf(\alpha)$-many blocks. The stages of the construction are carried out just as for the previous proof. Of course, action on behalf of the various requirements is different as described above.

The proof that the construction works is also very similar to that of Post's problem. As before an inductive argument is necessary to show that every block eventually settles down. Once this is shown the various properties of A and B are easily shown. For a more detailed and precise account of this argument see Shore [14] or [16].

4. Applications

The arguments we have discussed so far have been refined and applied to several other areas of recursion theory. The major area we will consider is admissible sets and structures, but before doing this I want to look briefly at two other places where these methods have been of use.

The first of these is Kleene recursion in an object of finite type. For the relevant definitions see Sacks' article in this volume or Kleene [9]. Harrington [7] was able to apply the results of α-recursion theory to solve a particular version of Post's problem for higher types. In order to do this Harrington used the following strong solution to Post's problem for α-recursion theory.

Theorem (Shore [17]): There exists a pair of integers m,n such that for all B and α where $\langle L_\alpha[B], \varepsilon, B\rangle$ is admissible, m and n are indices for α-B-r.e. hyperregular sets such that neither is α-B-recursive in the other.

The proof of this theorem is a refinement and extension of the blocking proof presented earlier. In particular, extra requirements are necessary to make sure the sets constructed are hyperregular. To make the construction uniform the parameters used in the construction, α^* and $\Sigma_2 cf(\alpha)$, cannot be used outright but must be "guessed at", that is approximated, during the construction.

Harrington's version of Post's problem considers reduction procedures which allow subconstructive parameters. These parameters are essentially ordinals which arise from computations in an object of type n + 2 with type n - 1, that is sub-individual, parameters. Harrington [7] shows that there are two r.e. sets of subconstructive ordinals which are incomparable with respect to the above reducibility.

The proof proceeds by reducing Post's problem in the above setting to Post's problem for an admissible structure of the form $\langle L_\alpha[B], \varepsilon, B\rangle$ and then applying Shore's result. Recently, a strong version of Post's problem for higher types has been proved by Sacks. The blocking method is also an ingredient of his proof. For a more detailed discussion of this see Sacks' article in this volume.

A second area of application for these methods has been to β-recursion theory where β is a weakly admissible ordinal (that is $\Sigma_1 cf(\beta) \geq \beta^*$). Post's problem was originally solved for weakly admissible ordinals β by Sy Friedman [4]. Friedman's proof was very close to the above argument for α-recursion theory. The one important difference is that requirements have to be added to insure that the two sets constructed are tamely-β-r.e.. The splitting theorem works similarly in this case, (see Homer [8]). It should be noted that the admissible collapse of Maass [10] can be used to get these results directly from the corresponding results of α-recursion theory.

Admissible Sets

In this section I want to present the axioms for admissible sets due to Kripke and Platek. Models for these axioms were envisioned as structures within which to do recursion theory. As such, it is natural to try to determine the solution to Post's problem in various admissible sets. In particular, it would be interesting to see if there are certain necessary and sufficient conditions for Post's problem to have a positive solution in an admissible set. Before addressing this question we need to briefly investigate how much recursion theory can be developed in these structures. For an in-depth look at admissible sets see Barwise [2].

<u>Definition</u>: An <u>admissible</u> <u>set</u> A is a transitive set satisfying

1. union: $x \in A \longrightarrow \bigcup x \in A$

2. pairing: $x,y \in A \longrightarrow \{x,y\} \in A$

3. Δ_0-comprehension: If $\phi(x)$ is any Δ_0 formula, $\forall a \exists b \forall x(x \in b \longleftrightarrow x \in a \wedge \phi(x))$.

4. Δ_0-bounding: if $\phi(x,y)$ is any Δ_0 formula, $(\forall x \exists y \phi(x,y)) \rightarrow (\forall s \exists t \forall x \in s \exists y \in t(\phi(x,y)))$.

There are many stronger forms of comprehension and bounding which are provable from these axioms. We now state one of these in order to relate these axioms to the replacement principle of α-recursion theory.

Theorem (Σ_1-replacement): If A is admissible, $a \in A$ and f is $\Sigma_1(A)$ then $f \restriction a$ is an element of A.

Hence we see that the same replacement principle is present in both α-recursion theory and the theory of admissible sets. The hope is that by studying this theory we will gain some understanding of exactly what properties are most important to carry out various recursion theoretic arguments. Studying admissible sets allows us to try to apply our methods to structures other than $L(\alpha)$ without giving up the strong closure properties provided by admissibility.

Typical examples of admissible sets are the collection of hereditarily finite sets, sets of hereditary cardinality less than κ, for any cardinal κ, and models of ZF. The following definition links up the notions of admissible set and admissible ordinal.

Definition: Let A be admissible. Then $|A|$ = ordinal height of A = least α(α is an ordinal and $\alpha \notin A$).

Claim: α is an admissible ordinal iff $\alpha = |A|$ for some admissible set A.

Sketch of proof: Assume α is an admissible ordinal. Since then $L(\alpha)$ satisfies Σ_1-replacement it is easy to check that $L(\alpha)$ satisfies the axioms for an admissible set and $|L(\alpha)| = \alpha$. Conversely assume $\alpha = |A|$ for some admissible A. Now carry out the construction of $L(\alpha)$ within A. By absoluteness, this is the real $L(\alpha)$ and by Σ_1-replacement in A, $L(\alpha)$ satisfies Σ_1-replacement. So α is an admissible ordinal. $\dashv$

We now turn to exploring recursion theory on admissible sets.

<u>Definition</u>: Let A be admissible.

1. $B \subseteq A$ is A-r.e. if B is $\Sigma_1(A)$.
2. $B \subseteq A$ is A-rec. if B is $\Delta_1(A)$.
3. $B \subseteq A$ is A-finite if $B \in A$.

<u>Theorem</u> (Enumeration Theorem): There is a universal A-r.e. relation $W(x,y)$ such that, if $W_a = \{y | A \models W(a,y)\}$, W_a ranges over all A-r.e. sets as a ranges over A.

<u>Corollary</u>: There is a non-A-recursive, A-r.e. set.

The above two facts are about as far as we can go without any additional assumptions on the admissible set. In particular, there is no hope of a positive solution to Post's problem for all admissible sets as the following theorem of Harrington shows.

<u>Theorem</u> (Harington [7]): There exists an admissible set A in which every A-r.e. set is $\Delta_1(A)$ or Σ_1-complete.

There are a couple of interesting open problems associated with this result. The admissible set which Harrington constructs has height λ'_2.

It is not known if Post's problem can fail in an admissible