S. Faedo (Ed.)

CIME Summer Schools

il principio di minimo e sue applicazioni alle equazioni funzionali

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FONDAZIONE CIME ROBERTO CONTI

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Lectures given at the
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IL PRINCIPIO DI MINIMO E SUE APPLICAZIONI ALLE EQUAZIONI FUNZIONALI

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ON ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

by L. NIRENBERG (New York) (*)

Outline.

This series of lectures will touch on a number of topics in the theory of elliptic differential equations. In Lecture I we discuss the fundamental solution for equations with constant coefficients. Lecture 2 is concerned with Calculus inequalities including the well known ones of Sobolev. In lectures 3 and 4 we present the Hilbert space approach to the Dirichlet problem for strongly elliptic systems, and describe various inequalities. Lectures 5 and 6 comprise a self contained proof of the well known fact that « weak » solutions of elliptic equations with sufficiently « smooth » coefficients are classical solutions.

In Lectures 7 and 8 we describe some work of Agmon, Douglis, Nirenberg [14] concerning estimates near the boundary for solutions of elliptic equations satisfying boundary conditions. This work is based on explicit formulas, given by Poisson kernels, for solutions of homogeneous elliptic equation with constant coefficients in a half space.

Throughout, for simplicity we treat one equation in one unknown. The material will on the whole be self contained, though of course not all proofs can be included. However, we shall attempt to indicate those of the main results.

(*) Questo ciclo di conferenze è stato tenuto a Pisa dal 1° al 10 settembre 1958, e ha fatto parte del corso del C. I. M. E. che ha avuto per tema: « Il principio di minimo e sue applicazioni alle equazioni funzionali ». Tale corso si è svolto in collaborazione con la Scuola Narmale Superiore e l'Istituto Matematico dell'Università di Pisa. In questi Annali saranno successivamente pubblicati i corsi di conferenze tenuti dai professori C. B. Morrey e L. Bers.

Lecture I. The Fundamental Solution.

I would like to start with a few general and somewhat unrelated comments. In studying differential equations one is usually interested in obtaining *unique* solutions by imposing suitable boundary or initial conditions, the kind depending on the so-called «type» of the equation - elliptic, hyperbolic, etc. However, the type classification for general equations has not been carried out, and in many cases it is not known what boundary conditions to impose. Indeed for equations that change type — and we are all familliar with the initial work in this field due to Professor Tricomi — the nature of the boundary conditions is far from obvious.

Thus if one considers an arbitrary equation without regard to type it is a natural question to ask whether there exist solutions at all. In fact there are occasions when one simply wants some solutions. Such occur often in differential geometry. Take a well known case: to introduce isothermal coordinates with respect to a given Riemannean metric on a two dimensional manifold. This reduces to a local problem of finding nontrivial solutions of a differential equation in a neighborhood of a point.

Another question is: are there solutions in the large of a given equation. For the preceding this is answered by uniformization theory for Riemann surfaces.

In this talk we will consider for some special cases the question: For a given differential operator L are there solutions of $Lu = f$ for « well behaved » functions f. Of course equations with analytic coefficients always have local solutions, obtained for instance by power series expansions (Cauchy-Kowalewski).

Recently Hans Lewy [1] exhibited an equation with C^∞ coefficients having no solutions ewen locally. Since it is easy to describe, we present it:

In 3-space with coordinates x, y, t, set $z = x + iy$, write the Cauchy-Riemann operator as $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, and consider the differential equation

$$Lu = \left(\frac{\partial}{\partial \bar{z}} + iz\frac{\partial}{\partial t}\right)u = \frac{\partial \psi(t)}{\partial t}$$

where the right hand side is a continuous real function of t alone which, for convenience, is written as a derivative of a real function ψ.

THEOREM: *If there is a continuously differentiable solution u of the equation in a neighborhood of the origin, then $\psi(t)$ is real analytic.*

Thus for any non-analytic ψ there is no solution near the origin. (The proof may be easily modified to show that there are also no « generalized solutions »).

Proof: If we integrate $\frac{\partial u}{\partial z} d\theta$ over a circle $|z|^2 = s \geq 0$, $z = s^{1/2} e^{i\theta}$, we establish easily the identity

$$\int_0^{2\pi} \frac{\partial u}{\partial z} d\theta = \frac{\partial}{\partial s} \int_0^{2\pi} z\, u\, d\theta .$$

Now set $\zeta = s + i t$ and $U(\zeta) = \int z\, u\, d\theta$. Integrating the equation for u over the circle we find that U satisfies

$$\left(\frac{\partial}{\partial s} + i \frac{\partial}{\partial t}\right) U = 2\pi \frac{d\psi}{dt}$$

or

$$\left(\frac{\partial}{\partial s} + i \frac{\partial}{\partial t}\right) (U + 2\pi i \psi) = 0 .$$

It follows that $V(\zeta) = U + 2\pi i \psi$ is a holomorphic function of $\zeta = s + it$ in a domain near the origin with re $\zeta = s > 0$. But on $s = 0$ the function U, i. e. the real part of V, vanishes, and therefore V can be continued analytically across $s = 0$. Hence ψ is analytic.

In [1] Lewy also constructs a function F such that the equation $Lu = F$ has no « smooth » solution in the neighborhood of any point. Lewy also conjectures that there are *homogeneous* equations with C^∞ coefficients having no solutions in the neighborhood of any point.

The simplest class of differential operators L of arbitrary type, for which one might expect solutions u of

$$Lu = f \tag{1.1}$$

to exist, for all well behaved functions f, are operators with constant coefficients. In the last few years a considerable study has been made of general differential operators with constant coefficients. (See Ehrenpreis [2], Hörmander [3], Malgrange [4]. Solutions of (1.1) can be found, at least locally, if one knows that a fundamental solution E of $LE = \delta$ (the Dirac δ function) exists. This is a (possibly generalized) function E such that

$$E * Lu = u$$

for all C^∞ functions u with compact support. We shall denote the class of such functions by C_0^∞. Here $*$ denotes convolution. Then if f is in C_0^∞ the function $u = E * f$ is a solution of (1.1).

Malgrange [4] and Ehrenpreis [2] proved the existence of a fundamental solution for any differential operator with constant coefficients. However it is not difficult to construct one explicitily, as Hörmander, and also Trêves [5], have shown, and we shall now describe such a construction.

First we fix our

NOTATION: We consider functions $u(x)$ of n variables $x = (x_1, \dots, x_n)$ and denote the differentiation vector by $D = (D_1, \dots, D_n)$, $D_i = \partial/\partial x_i$. The letters β, γ, μ, ν will denote vectors $\beta = (\beta_1, \dots, \beta_n)$ with non-negative integral coefficients β_i, and we set $|\beta| = \Sigma \beta_i$. Otherwise for any vector $\xi = (\xi_1, \dots, \xi_n)$, $|\xi|$ will represent its Euclidean length $|\xi|^2 = \Sigma |\xi_i|^2$, and $\xi \cdot \eta = \Sigma \xi_i \eta_i$. We write

$$\xi^\beta = \xi_1^{\beta_1} \dots \xi_n^{\beta_n}, \qquad D^\beta = D_1^{\beta_1} \dots D_n^{\beta_n};$$

for convenience we shall also, on occasion, express a general m^{th} order partial derivative of a function u by $D^m u \cdot C_0^\infty$ will denote the class of C^∞ functions with compact support.

We consider now a differential operator L of order k with constant coefficients, which we may write as a polynomial in D of order k.

$$L = L(D).$$

In constructing the fundamental solution let us first argue in a heuristic manner. Introduce the Fourier transform of the function $u(x)$

$$\tilde{u}(\xi) = \int e^{-ix\cdot\xi} u(x)\, dx,$$

integration being over the entire n-space. Then

$$\widetilde{L(D)u} = L(i\xi)\, \tilde{u}(\xi).$$

So if $u = E * Lu = \int E(x-y)\, Lu(y)\, dy$ then

$$\tilde{u} = \tilde{E}(\xi)\, L(i\xi)\, \tilde{u}(\xi)$$

or

$$\tilde{E} = \frac{1}{L(i\xi)},$$

or

$$E(x) = (2\pi)^{-n} \int \frac{e^{ix\cdot\xi}}{L(i\xi)}\, d\xi . \tag{1.2}$$

Problem : give formula (1.2) *a meaning.*

In attempting to do this (and there are many ways) there are two difficulties that occur. The first is the non-integrability at infinity, duo to the fact that we are integrating over the full n-space. The second difficulty is caused by the real roots ξ of the polynomial $L(i\xi)$.

The first difficulty is easily overcome. It essentially expresses the fact that is general E is a distribution, i. e. a finite derivative of a continuous function. Instead of constructing E directly we shall construct the fundamental solution E_N of the operator $(1-\Delta)^N L = (1 - \sum_i D_i^2)^N L(D)$. We shall construct a fundamental solution E_N having continuous derivatives up to any given order, by taking N sufficiently large. We may then take, in the distribution sense,

$$E = (1-\Delta)^N E_N , \tag{1.3}$$

i. e. for f in C_0^∞ the function

$$u = E_N * (1-\Delta)^N f$$

is a solution of $Lu = f$.

Thus we consider, for $p(\xi) = 1 + \Sigma\, \xi_j^2$

$$E_N = (2\pi)^{-n} \int \frac{e^{ix\cdot\xi}}{p^N(\xi)\, L(i\xi)}\, d\xi . \tag{1.4}$$

Taking N large eliminates the first difficulty, i. e. the trouble at infinity.

Now to handle the second difficulty. We may assume, after a possible rotation of coordinates, that the coefficient of D_n^k in $L(D)$ is $\neq 0$, say unity. Consider $L(i\xi)$ as a polynomial in ξ_n. We shall first integrate in (1.4) with respect to the variable ξ_n, keeping $\xi' = (\xi_1, \dots, \xi_{n-1})$ fixed, however we shall move the line of integration from the real line to a parallel line lying in the complex ξ_n plane.

For fixed real ξ' there are k roots ξ_n of $L(i\xi)$. In the strip $|\mathfrak{Im}\,\xi_n| \leq \frac{1}{2}$ in the complex ξ_n plane there is therefore a line parallel to the real axis whose distance from any root is at least $(2k+2)^{-1}$, as one easily sees. Let us a choose one such line $\mathfrak{Im}\,\xi_n = c(\xi')$ whose distance to any root is at least $(4k+4)^{-1}$. The choice of $c(\xi')$ depends on ξ', but it is easy to see that $c = c(\xi')$ may be chosen so as to be continuous except on a set of ξ' of $(n-1)$-dimensional measure zero.

Setting $\eta = \eta(\xi') = (0, \ldots, c(\xi'))$ we now take as definition

$$E_N = (2\pi)^{-n} \int \frac{e^{ix\cdot(\xi+i\eta(\xi'))}}{p^N(\xi+i\eta)\, L(i(\xi+i\eta))}\, d\xi \tag{1.4$'$}$$

where integration is first with respect to ξ_n.

Since

$$|p(\xi+i\eta(\xi'))| \geq \frac{3}{4} \quad \text{and} \quad |L(i(\xi+i\eta))| \geq (4k+4)^{-k}$$

we see that E_N has derivatives up to any given order, if N is large enough.

We have finally to verify that for $u \in C_0^\infty$

$$u = E_N * (1-\Delta)^N Lu \equiv \int E_N(x-y)\,(1-\Delta)^N Lu(y)\, dy.$$

Setting $(1-\Delta)^N L(D) = L_N(D)$, the right hand side equals

$$(2\pi)^{-n} \iint \frac{e^{i(x-y)\cdot(\xi+i\eta)}}{L_N(i(\xi+i\eta))}\, d\xi\, L_N(D)\, u(y)\, dy.$$

Since u has compact support its Fourier transform $\tilde{u}(\xi)$ can be extended to complex vectors ξ as an entire analytic function, and since $u \in C^\infty$ the derivatives of $\tilde{u}$ die down faster that any power of $|\xi|$ as we go to infinity in a strip $|\Im m\, \xi| <$ constant. Thus, interchanging the order of integration in the above, we find that it equals

$$(2\pi)^{-n} \int \frac{e^{ix(\xi+i\eta)}}{L_N(i(\xi+i\eta))}\, L_N(i(\xi+i\eta))\, \tilde{u}(\xi+i\eta)\, d\xi =$$

$$= (2\pi)^{-n} \int e^{ix(\xi+i\eta)}\, \tilde{u}(\xi+i\eta)\, d\xi.$$

Because of the behaviour of $\tilde{u}$ of infinity we may shift the line of integration of the ξ_n parallel to itself and find that this expression

$$= (2\pi)^{-n} \int e^{ix\cdot\xi}\, \tilde{u}(\xi)\, d\xi = u(x).$$

Thus the function E_N defined by (1.4)′ is a fundamental solution for the operator L_N. The desired fundamental solution of Lu then is given by (1.3).

One sees easily that the fundamental solution E_N given by (1.4)′ has exponential growth in the x_n variable.

For further important work on fundamental solutions for equations with constant coefficients we refer to Hörmander [6].

Consider now elliptic differential operators with constant coefficients. These are operators L whose leading part L' — consisting of the terms of highest order — satisfy

$$L'(\xi) \neq 0 \quad \text{for real} \quad \xi \neq 0 .$$

We shall have need later of the fundamental solution for a homogeneous elliptic operator with constant coefficients, i. e. $L' = L$. For such, of course, the fundamental solution first constructed by Herglotz is well behaved at infinity. We shall use the following form of it, given in F. John's book [7].

$$E(x) = -\frac{1}{(2\pi i)^n (k+q)!} \Delta^{\frac{n+q}{2}} \int\limits_{|\xi|=1} \frac{(x \cdot \xi)^{k+q}}{L(\xi)} \log \frac{x \cdot \xi}{i} \, d\omega_\xi \tag{1.5}$$

where integration is over the full unit sphere with $d\omega_\xi$ as the element of area, q is a non-negative integer of the same parity n, i. e. $q+n$ is even, and the principal branch of the logarithm is taken with the plane slit along the negative real axis.

From (1.5) we obtain as a special case, for $L = \Delta$ power, the following identity which is due to F. John and used extensively in [7], representing the δ function in terms of plane waves: For u in C_0^∞

$$u = -\frac{1}{(2\pi i)^n q!} \Delta^{\frac{n+q}{2}} \left[\int\limits_{|\xi|=1} (x \cdot \xi)^q \log \frac{x \cdot \xi}{i} \, d\omega_\xi * u \right] . \tag{1.6}$$

In [7] John derives (1.6) from the known expression for the fundamental solution for a power of the Laplacean, and then derives (1.5) from (1.6). This may be done as follows. Suppose $K(x \cdot \xi)$ satisfies

$$L\,K(x \cdot \xi) = (x \cdot \xi)^q \log \frac{x \cdot \xi}{i} ,$$

then a fundamental solution of the operator L is given by

$$-\frac{1}{(2\pi i)^n q!} \Delta^{\frac{n+q}{2}} \int\limits_{|\xi|=1} K(x \cdot \xi) \, d\omega_\xi .$$

But such a K is easily found. If we set $x \cdot \xi = \sigma$ then $K(\sigma)$ satisfies

$$L(\xi)\left(\frac{d}{d\sigma}\right)^k K(\sigma) = \sigma^q \log \sigma / i ,$$

a solution of which is

$$K(\sigma) = \frac{1}{L(\xi)} \frac{q!}{(k+q)!} \sigma^{k+q}\left(\log \frac{\sigma}{i} + c_{k,q}\right),$$

with $c_{k,q}$ an appropriate constant. If we insert this into the above expression for the fundamental solution of L we obtain the expression

$$-\frac{1}{(2\pi i)^n (k+q)!} \Delta^{\frac{n+q}{2}} \int\limits_{|\xi|=1} \frac{(x\cdot\xi)^{k+q}}{L(\xi)}\left(\log\frac{x\cdot\xi}{i} + c_{k,q}\right)$$

which differs from (1.5), only by the term involving $c_{k,q}$. But this term is a polynomial of degree $k-n$ which is therefore a solution of $Lv=0$, and so may be ignored.

It should also be possible to derive (1.5) from the heuristic formula (1.2). (1.5) aserts that

$$-\frac{1}{(2\pi i)^n (k+q)!} \int\limits_{|\xi|=1} \frac{(x\cdot\xi)^{k+q}}{L(\xi)} \log\frac{x\cdot\xi}{i}\, d\omega_\xi \tag{1.7}$$

is a fundamental solution for the operator $\Delta^{\frac{n+q}{2}} L$. Let us attempt to derive this expression from the corresponding expression of (1.2):

$$(-1)^{\frac{n+q}{2}} (2\pi)^{-n} \int \frac{e^{ix\cdot\xi}}{|\xi|^{n+q} L(i\xi)}\, d\xi . \tag{1.8}$$

Arguing heurisitically again let us modify the expression by introducing polar coordinates in the ξ space

$$\xi = \varrho\eta , \qquad \varrho = |\xi| , \qquad |\eta| = 1 .$$

Then (1.8) becomes

$$(-1)^{n+q+k} (2\pi)^{-n} \int\limits_{|\eta|=1} \int\limits_0^\infty \frac{e^{i\varrho x\cdot\eta}}{L(\eta)} \varrho^{-1-q-k}\, d\varrho\, d\omega_\eta . \tag{1.8$'$}$$

Let us now write the heuristic expression

$$\int_0^\infty e^{i\varrho x\cdot\eta}\,\varrho^{-1-q-k}\,d\,\varrho \tag{1.9}$$

as a well defined contour integral

$$\frac{1}{2\pi i}\int_{\mathfrak{C}} e^{i\varrho x\cdot\eta}\,\varrho^{-1-q-k}\,(\log(-\varrho)+c) \tag{1.9$'$}$$

where the contour $\mathfrak{C}$ is a curve which goes from $+\infty$ in the complex ϱ plane, encircles the origin counterclockwise and returns to $+\infty$ along the real axis, the branch for the logarithm is the same as above, and the constant c is chosen so that

$$\int_{\mathfrak{C}} e^{i\varrho}\,\varrho^{-1-q-k}\left(\log(-\varrho)+c-\frac{i\pi}{2}\right)d\varrho = 0\,.$$

The expressions (1.9)$'$ may be evaluated explicity, and on insertion into (1.8)$'$, yields the expression (1.7). We leave the calculation to the reader.

Lecture II. Calculus Inequalities.

A priori estimates play a central role in the theory of partial differential equations. They are of various kinds — pointwise estimates for derivatives of solutions and their modulus of continuity, and estimates of, say, L_p norms of solutions and their derivatives — and it is naturally important to understand the relationships between these various estimates.

For instance, the well known results of Sobolev assert that if the m'th order derivatives $D^m u$ of a function $u(x_1, \dots, x_n)$ (with compact support) are in L_r, $1 < r < \infty$ then lower order derivatives $D^j u$, $j < m$ belong to L_p for some p, or, if r is sufficiently high, the $D^j u$ are bounded and satisfy a Hölder condition with a certain exponent α.

Since we shall often make use of it, let us recall here the notion of

Hölder continuity. A function $f(x)$ defined on a set S in a Euclidean space satisfies a Hölder condition there with exponent α, $0 < \alpha < 1$, if

$$[f]_\alpha = [f]_\alpha^S = \underset{x,y\in S}{e\,.\,u\,.\,b}\;\frac{|f(x)-f(y)|}{|x-y|^\alpha} \tag{2.1}$$

is finite. It is Hölder continuous (exponent α) in a domain if it satisfies a Hölder condition with exponent α in every cempact subset of the domain.

This lecture is concerned with calculus inequalities relating integral and pointwise estimates of functions and their derivatives. The recent important result of de Giorgi [11] on the differentiability of solutions of regular variational problems seems in fact to be based on a calculus inequality asserting that certain integral estimates imply Hölder continuity. We shall consider functions $u(x)$ defined in n-dimensional Euclidean space and belonging to L_q, and whose derivatives of order m belong to L_r, $1 \leq q, r \leq \infty$. We shall present interpolative inequalities for the L_p and Hölder norms $[\ ]_a$ of derivatives $D^j u$, $0 \leq j < m$, for the maximal range of p and α. Our inequalities are a combination of, and include, those usually called of Sobolev type (which hold also for fractional derivatives, and rather straightforward proofs of which may be found in [8]), and familiar interpolative inequalities such as

$$M_1^2 \leq \text{constant } M_0 \cdot M_2$$

where M_i is $e.u.b.$ of the L_p norms of the derivatives of order i of a function u, $i = 0, 1, 2$. The proofs use only first principles and are entirely elementaty. (No attempt will be made here to obtain best constants). The inequalities is this section were presented at the Int'e Congress in Edinburgh August 1958, where we learned that almost equivalent results had also been proved by E. Gagliardo.

In this lecture we shall use the following

Notation: For $-\infty < \frac{1}{p} < \infty$ we defins the norms and seminorms $|u|_p$ for functions $u(x)$ defined in a domain $\mathcal{D}$ in n-dimensional spaces:

For $p > 0$

$$|u|_p = \text{the } L_p \text{ norm of } u \text{ in } \mathcal{D}.$$

$$= \left(\int_{\mathcal{D}} |u|^p \, dx \right)^{\frac{1}{p}}.$$

For $p < 0$ set $s = [-n/p]$, $-\alpha = s + n/p$ and define

$$|u|_p = e.u.b.[D^s u]_\alpha^{\mathcal{D}} \qquad \text{if } \alpha > 0,$$

$$|u|_p = e.u.b.|D^s u| \qquad \text{if } \alpha = 0,$$

where $e.u.b.$ is taken with respect to all partial derivatives D^s of order s, and over points in $\mathcal{D}$.

We define $|D^j u|_p$ as the maximum of the $|\ |_p$ norms of all j-th order derivatives of u.

We shall express our result for functions u defined in the entire n-space E^n. Extension to other domains will be described briefly in the remarks after the theorem.

THEOREM: *Let u belong to L_q in E^n and its derivatives of order m, $D^m u$, belong to L_r, $1 \leq q, r \leq \infty$. For the derivatives $D^j u$, $0 \leq j < m$, the following inequalities hold*

$$|D^j u|_p \leq \text{constant}\, |D^m u|_r^a\, |u|_q^{1-a}, \tag{2.2}$$

where

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q},$$

for all a in the interval

$$\frac{j}{m} \leq a \leq 1 \tag{2.3}$$

(*the constant depending only on n, m, j, q, r, a*), *with the following exceptional cases*

1. *If $j = 0$, $rm < n$, $q = \infty$ then we make the additional assumption that either u tends to zero at infinity or $u \in L_{\tilde{q}}$ for some finite $\tilde{q} > 0$.*

2. *If $1 < r < \infty$, and $m - j - n/r$ is a non negative integer then* (2.2) *holds only for a satisfying $j/m \leq a < 1$.*

We shall not give a complete proof of the theorem here but shall indicate the main steps. First some comments.

1. The value of p is determined simply by dimensional analysis.

2. For $a = 1$ the fact that u is contained in L_q does not enter in the estimate (2.2), and the estimate is equivalent to the results of Sobolev (note that we permit r to be unity).

3. That j/m is the smallest possible value for a may be seen by taking $u = \sin \lambda x_1 \zeta(x)$ where ζ is in C_0^∞: For large λ we have $|u|_q = 0(1)$, $|D^j u|_p = 0(\lambda^j)$, $|D^m u|_r = 0(\lambda^m)$ where no 0 can be replaced by o.

4. It will be clear from the proof that the result holds also for u defined in a product domain

$$-\infty < x_s < \infty,\ 0 < x_t < \infty\ :\ s = 1, \dots, k:\ t = k+1, \dots, n,$$

and hence for any domain that can be mapped in a one-to-one way onto such a domain by a sufficiently «nice» mapping.

5. For a bounded domain (with «smooth» boundary) the result holds if we add to the right side of (2.1) the term

$$\text{constant} \, | u |_{\tilde{q}} .$$

for any $\tilde{q} > 0$. The constants then depend also on the domain.

6. Similar estimates hold for the L_p norms of $D^j u$ on linear subspaces of lower dimension, for suitable p.

7. Similar interpolation inequalities also hold for fractional derivatives, but their proof is not so elementary.

The theorem, in its full generality should be useful in treating nonlinear problems. We mention in particular that from (2.2) for $a = j/m$, $q = \infty$ it follows that the set of functions u which are bounded and have derivatives of order m belonging to L_r forms a Banach Algebra. For $r = 2$ this is called the Schauder ring.

The proof of the theorem is elementary and contains in particular an elementary proof for the Sobolev case $a = 1$. In order to prove (2.2) for any given j one has only to prove it for the extreme values of a, j/m and unity. (If Case 2 holds some additional remark has to be made.) For in general there is a simple

Interpolation Lemma : if $-\infty < \lambda \leq \mu \leq \nu < \infty$ *then*

$$| u |_{\frac{1}{\mu}} \leq c \, | u |_{\frac{1}{\lambda}}^{\frac{\nu-\mu}{\nu-\lambda}} \cdot | u |_{\frac{1}{\nu}}^{\frac{\mu-\lambda}{\nu-\lambda}}$$

where c is independent of u.

The lemma is easily proved; for $\lambda > 0$ it is merely the usual interpolation ineqnality for L_p norms.

Let us turn now to the proof of the theorem, or at least to the main points. Consider first the Sobolev case, $a = 1$. It suffices to consider the case $j = 0$, $m = 1$, from which the general result may then be derived. If $r > n$ (2.2) asserts that u satisfies a certain Hölder condition, and an elementary proof due to Morrey has long been known. We shall sketch it here for functions defined in a general domain $\mathcal{D}$.

Definition : A domain $\mathcal{D}$ is said to have the strong cone property if there exist positive constants d, λ and a closed solid right spherical cone V of fixed opening and height such that any points P, Q in $\overline{\mathcal{D}}$ (the closure of $\mathcal{D}$) with

$$| P = Q | \leq d$$

are vertices of cones V_P, V_Q lying in $\bar{\mathcal{D}}$ which are conrguent to V and have the following property: the volume of the intersection of the sets: V_P, V_Q, and the two spheres with conters P, Q and radius $|P-Q|$, is not less than $\lambda\,|P-Q|^n$.

We now prove the assertion

If u has first derivatives in L_r, $r > n$, in a domain $\mathcal{D}$ having the strong cone property, then for points P, Q in $\mathcal{D}$ with $|P-Q| \leq d$, we have

$$\frac{|u(P)-u(Q)|}{|P-Q|^{1-\frac{n}{r}}} \leq \text{constant}\,|D u|_r$$

where the constant depends only on d, λ, V, n and r.

(From this follows easily an estimate for $[u]_{1-\frac{n}{r}}$, depending on the domain).

Proof: Set $s = |P-Q|$ and let $S_P\,(S_Q)$ be the intersection of $V_P\,(V_Q)$ with the sphere about $P(Q)$ radius s. Set $S_P \cap S_Q = S$. If R is a point in S we have, on integrating with respect to R over S,

$$\text{Volume of } S \cdot |u(P)-u(Q)| \leq \int_S |u(P)-u(R)|\,dR + \\ + \int_S |u(R)-u(Q)|\,dR.$$

Because of the strong cone property the left hand side is not less than

$$\lambda\, s^n\, |u(P)-u(Q)|.$$

The first term on the right may be estimated as follows. Introducing polar coordinates ϱ, η, about P, where η is a unit vector, we find easily that the first term in the right is bounded by

$$\int_{S_P} \varrho^{n-1}\, d\,\omega_\eta\, d\,\varrho \int_0^\varrho \left|\frac{\partial u}{\partial \varrho}\right| d\,\varrho \leq \text{constant}\ s^n \int_{S_P} \left|\frac{\partial u}{\partial \varrho}\right| \frac{d\,x}{\varrho^{n-1}}$$

(where $d\,\omega$ is the element of area on the unit sphere, and $d\,x$ is the element of volume)

$$\leq \text{constant}\ s^n \left(\int_{S_P} \left|\frac{\partial u}{\partial \varrho}\right|^r d\,x\right)^{\frac{1}{r}} \left(\int_{S_P} \varrho^{(1-n)\frac{r}{r-1}}\, d\,x\right)^{\frac{r-1}{r}}$$