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EQUAZIONI DIFFERENZIALI ASTRATTE

CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C. I. M. E.)

T O S I O K A T O

SEMI-GROUPS AND TEMPORALLY INHOMOGENOUS EVOLUTION EQUATIONS

ROMA - Istituto Matematico dell'Università

SEMI-GROUPS AND TEMPORALLY INHOMOGENOUS EVOLUTION EQUATIONS

by T. KATO

INTRODUCTION

These lectures are concerned with the Cauchy problem for the time-indépendent evolution equation

$$\text{(E)} \qquad \frac{du}{dt} + A(t)u = f(t)\ , \qquad 0 < t \leq T\ ; \qquad u(o) = u_o.$$

The unknown $u = u(t)$ and the given function $f(t)$ take values in a Banach space X ; $A(t)$ is a (in general unbounded) linear operator in X depending on t .

It will suffice to mention here only a few examples of (E).

Ex. 1. A parabolic differential equation

$$\frac{\partial u}{\partial t} - \sum_{j,k=1}^{n} a_{jk}(x,t)\frac{\partial^2 u}{\partial x_j \partial x_k} - \sum_{j=1}^{n} a_j(x,t)\frac{\partial u}{\partial x_j} - a(x,t)u = f(x,t)$$

$$x \in \Omega \subset R^n\ , \qquad 0 < t \leq T$$

is in the form (E) with an obvious definition of A(t). The boundary conditions, which may depend on t , are included in the definition of A(t).

Ex. 2. The Schrödinger equation

$$\frac{1}{i}\frac{\partial u}{\partial t} + \Delta\ u - V(x,t)u = 0\ , \qquad x \in R^3$$

also has the form (E). Here $A(t) = i(\Delta - V(.\ ,t))$ is i times a self-adjoint operator (at least formally) in $X = L^2(R^3)$.

Ex. 3. The wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta\ u$$

in $x \in R^3$ may be reduced to the form (E) by writing

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} - \begin{pmatrix} 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1} & 0 & 0 & 0 \\ \frac{\partial}{\partial x_2} & 0 & 0 & 0 \\ \frac{\partial}{\partial x_3} & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

where v_1, v_2, v_3 are auxiliary functions. This has the form (E), where u is replaced by the 4-component vector function (u, v_1, v_2, v_3).

In what follows I want to deduce several sufficient conditions on $A(t)$ and $f(t)$ in order that (E) has a unique solution.

§ 1. GENERATION OF DIFFERENT TYPES OF SEMI-GROUPS

1. Let us consider (E) first in the special case when $A(t) = A$ is independent of t :

$$(E_o) \qquad \frac{du}{dt} + Au = f(t), \qquad u(o) = u_o .$$

The solution is formally given by

$$(S_o) \qquad u(t) = e^{-tA}u_o + \int_o^t e^{-(t-s)A}f(s)ds .$$

The problem is, therefore, essentially that of constructing the exponential function e^{-tA}. This is exactly the problem of generating a semi-group $\{ e^{-tA} \}$ from a given operator A .

In what follows we consider only strongly continuous semi-groups on $[0, \infty)$. Thus $\{ e^{-tA} \}$ is a semi-group if and only if e^{-tA} is strongly continuous for $0 \leq t < \infty$ with $e^{-0A} = 1$ and has the semi-group property : $e^{-tA}e^{-sA} = e^{-(t+s)A}$ for all s, $t \geqslant 0$.

The generation of such semi-groups has been discussed in detail in Professor Phillips' lectures. We reproduce here some of his results that we need, with several additional remarks.

Definition 1. 1. We say $-A \in (Bo)$ if

1) A is densely defined and closed.

2) any $\lambda > o$ belongs to the resolvent set $\rho(-A)$ of $-A$, with

$$| (\lambda + A)^{-n} | \leqslant \frac{M}{\lambda^n}, \qquad n = 1, 2, 3, \ldots .$$

where M is a constant independent of λ or n .

Theorem 1. 2. Let $-A \in (Bo)$. Then there exists a unique semi-group $\{ e^{-tA} \}$ with $| e^{-tA} | \leqslant M$ such that

$$(D) \qquad \frac{d}{dt} e^{-tA}u = -Ae^{-tA}u = - e^{-tA}Au$$

for $u \in D_{A'}$ e^{-tA} commutes with $(\lambda + A)^{-1}$.

Proof. See Phillips' lectures

Remark 1. 3. If $-A \in$ (Bo), it follows that all complex λ with $\mathrm{Re}\,\lambda > 0$ belong to $\rho(-A)$, with

$$\left|(\lambda + A)^{-n}\right| \leq \frac{M}{(\mathrm{Re}\,\lambda)^n} \qquad n = 1, 2, \ldots .$$

This is seen by considering the Laplace transforms of $t^{n-1} \cdot e^{-tA}$ (see Phillips).

Definition 1. 4. If M - 1 above, $\{e^{-tA}\}$ is called a contraction semi-group. The subset of (Bo) determined by M = 1 will be denoted by (Co). (Note that $M \geq 1$ in general).

2. We now introduce another subset of (Bo) which is important in our problems.

Definition 1. 5. We say $-A \in$ (Ho) if

1) A is densely defined and closed

2) The spectrum $\sigma(A)$ of A is a subset of a sector

$|\arg \lambda| \leq \frac{\pi}{2} - \omega$, $\omega > 0$, and

$$\left|(\lambda + A)^{-1}\right| \leq \frac{M_\varepsilon}{|\lambda|} \qquad \text{for} \qquad |\arg \lambda| \leq \frac{\pi}{2} + \omega - \varepsilon$$

Remark 1. 6. (Ho) $\subset$ (Bo). This is not obvious from the definition, but follows from Theorem 1. 7. below (again take the Laplace transforms of $t^{n-1} e^{-tA}$).

Theorem 1. 7. Let $-A \in$ (Ho). Then there exists a unique semi-group $\{e^{-tA}\}$ such that for $t > 0$

$$\text{(D')} \qquad -e^{-tA}A \supset \frac{d}{dt} e^{-tA} = -A\, e^{-tA} \in B \,^{1)}.$$

---

(1) We denote by B the set of all bounded linear operator in X with domain X.

$\{e^{-tA}\}$ can be continued analytically to the sector $|\arg t| < \omega$, $t \neq 0$, with (D') preserved. Furthermore, e^{-tA} and tAe^{-tA} are uniformly bounded in any smaller sector :

$$|e^{-tA}| \leq M'_\varepsilon \quad , \quad |t|\,|Ae^{-tA}| \leq M'_\varepsilon \quad , \quad |\arg t| \leq \omega - \varepsilon \quad ,$$

and $e^{-tA} \longrightarrow 1$ strongly when $t \longrightarrow 0$ in this smaller sector [1].

Proof. We can define e^{-tA} by the Dunford integral :

$$\text{(1)} \qquad e^{-tA} = \frac{1}{2\pi i} \int_C e^{\lambda t} (\lambda + A)^{-1} d\lambda \in \mathcal{B} \quad , \qquad t > 0 ,$$

where C is a curve, running in $\rho(-A)$, from $\infty e^{-i\vartheta}$ to $\infty e^{i\vartheta}$ where $\frac{\pi}{2} < \vartheta < \frac{\pi}{2} + \omega$. Thus the integral is absolutely convergent and defines an operator of $\mathcal{B}$.

The semi-group property follows from the standard argument with Dunford integrals. We have namely

$$\text{(2)} \qquad e^{-sA} = \frac{1}{2\pi i} \int_{C'} e^{\lambda' s} (\lambda' + A)^{-1} d\lambda' \quad ,$$

where C' is obtained from C by a slight shift to right. Multiplying (1), (2) and using the resolved equation, we have

$$e^{-tA} e^{-sA} = \left(\frac{1}{2\pi i}\right)^2 \int_C \int_{C'} e^{\lambda t + \lambda' s} \frac{1}{\lambda' - \lambda} \left[(\lambda + A)^{-1} - (\lambda' + A)^{-1} \right] d\lambda \, d\lambda' ,$$

where the order of integration is arbitrary.

$$\text{Now} \quad \int_C e^{\lambda t} \frac{d\lambda}{\lambda' - \lambda} = 0 \quad \text{and} \quad \int_{C'} \frac{e^{\lambda' s}}{\lambda' - \lambda} d\lambda' = 2\pi i \, e^{\lambda s}$$

since C lies to left of C'. Hence

$$e^{-tA} e^{-sA} = \frac{1}{2\pi i} \int_C e^{\lambda (t+s)} (\lambda + A)^{-1} d\lambda = e^{-(t+s)A} ,$$

---

(1) For analytic semi-groups, see [15] as well as the book by Hille-Phillips.

proving the semi-group property of $\{e^{-tA}\}$.

That e^{-tA} has an analytic continuation is obvious from (1). In fact the integral of (1) converges for any t with $|\arg t| < \vartheta - \frac{\pi}{2}$, hence for any t with $|\arg t| < \omega$, by taking ϑ suitably. Moreover,

$$(3) \qquad \frac{d}{dt} e^{-tA} = \frac{1}{2\pi i} \int_C \lambda e^{\lambda t} (\lambda + A)^{-1} d\lambda \in B, \qquad t \neq 0.$$

Since $\lambda(\lambda + A)^{-1} = 1 - A(\lambda + A)^{-1}$ and $\int_C \lambda e^{\lambda t} d\lambda = 0$, (D') follows from (3) (note that $A(\lambda + A)^{-1} A \subset A(\lambda + A)^{-1}$).

To prove the uniform boundedness of e^{-tA}, we change the integration variable from λ to $\lambda' = \lambda t$ in (1). The corresponding integration path tC can be deformed to a path C', independent of t, which runs from $\infty e^{-i\vartheta'}$ to $\infty e^{ti\vartheta'}$ with $\vartheta' = \frac{\pi}{2} + \varepsilon$, $\varepsilon > 0$ being very small.

The resulting expression

$$e^{-tA} = \frac{1}{2\pi i} \int_{C'} e^{\lambda'} \left(\frac{\lambda'}{t} + A\right)^{-1} \frac{d\lambda'}{t}$$

is true for any $t \neq 0$ with $|\arg t| \leq \omega - \varepsilon$. Since

$\left|\left(\frac{\lambda'}{t} + A\right)^{-1}\right| \leq M / \left|\frac{\lambda'}{t}\right| = M|t| / |\lambda'|$, it follows that

$$(3)' \qquad |e^{-tA}| \leq \frac{M}{2\pi} \int_{C'} |e^{\lambda'}| \frac{|d\lambda'|}{|\lambda'|} = M'_\varepsilon .$$

In the same way one proves $|Ae^{-tA}| \leq M''_\varepsilon / |t|$.

To prove $e^{-tA} \to 1$, $t \to 0$, we note that

$$(e^{-tA} - 1)u = \frac{1}{2\pi i} \int_C e^{\lambda t} \left[(\lambda + A)^{-1} - \lambda^{-1}\right] u \, d\lambda = \frac{-1}{2\pi i} \int e^{\lambda t} (\lambda + A)^{-1} A u \frac{d\lambda}{\lambda}$$

if $u \in D_A$. Hence for $t \to 0$

$$(e^{-tA} - 1)u \to -\frac{1}{2\pi i} \int (\lambda + A)^{-1} A u \frac{d\lambda}{\lambda} = 0$$

(the integrand is $O(\lambda^{-2})$ for $\lambda \to \infty$, $\mathrm{Re}\, \lambda > 0$).

Since e^{-tA} is uniformly bounded for $|\arg t| \leq \omega - \varepsilon$ as proved above and since D_A is dense, this proves that $e^{-tA} \to 1$ strongly, q. e. d.[1).]

Remark 1. 8. Theorem 1. 7. implies that, if $-A \in (H0)$, e^{-tA} sends X into D_A and $e^{-tA}u$ is always differentiable for any $u \in X$, if $t \neq 0$. This is a great difference from the case $-A \in (B0)$, where $e^{-tA}u \in D_A$ is in general expected only for $u \in D_A$.

Remark 1. 9. There are many examples of operators of (H0). Generally speaking, any strongly elliptic partial differential operator with "ordinary" boundary conditions belongs to (H0). Furthermore if X is a Hilbert space, there is a rather general sufficient condition for $-A \in (H0)$. Suppose that the numerical range $N_A = \{ (Au, u) \mid |u| = 1 , u \in D_A \}$ of A is a subset of a sector $|\arg \lambda| \leq \frac{\pi}{2} - \omega$, $\omega > 0$. If, in addition, there is at least one point λ exterior to N_A that belongs to $\rho(A)$, then $-A \in (H0)$[2).]

3. We now consider the solution of the inhomogeneous equation (E_0).

Definition 1. 10. By a solution of (E) we mean a function $u(t)$ with the following properties.

1) $u(t)$ is (strongly) continuous for $0 \leq t \leq T$, $u(o) = u_0$.
2) $u(t)$ is (strongly) differentiable for $0 < t \leq T$.
3) $u(t) \in D_{A(t)}$ for $0 < t \leq T$ so that $A(t)u(t)$ makes sense.
4) (E) is true for $0 < t \leq T$.

The same definition applies to (E_0) when $A(t) = A$ is constant.

Theorem 1. 11. Let $-A \in (B0)$. Then any solution of (E_0) is given by (S_0) if $f(t)$ is continuous for $0 \leq t \leq T$. Conversely, $u(t)$ given by (S_0) is a solution of (E_0) if $u_0 \in D_A$ and $f(t)$ is continuously differentiable. In this ca-

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(1) The uniqueness of $\{e^{-tA}\}$ with the properties stated follows from Theorem 1. 2.

(2) This is due to the fact that $|(\lambda + A)u| \geq d_\lambda |u|$ for any $u \in D_A$ where d_λ is the distance of λ from N_{-A}. It follows, under the condition stated, that $|(\lambda + A)^{-1}| \leq 1/d_\lambda \leq M/|\lambda|$.

se $Au(t)$ and $du(t)/dt$ are continuous [1].

<u>Proof.</u> Let $u(t)$ be a solution of (E_o). Then

$$\frac{d}{ds} e^{-(t-s)A} u(s) = e^{-(t-s)A} Au(s) + e^{-(t-s)A}(-Au(s)+f(s)) =$$
$$= e^{-(t-s)} f(s)$$

since $u(s) \in D_A$ and $e^{-(t-s)A} \in B$. Integration on s then gives (S_o) immediately.

Conversely, suppose $u_o \in D_A$ and $f(t)$ is continuously differentiable. Since $e^{-tA} u_o$ satisfies the homogeneous equation and the initial condition by Theorem 1. 2, we need only to consider the second term of (S_o). In other words, we may assume $u_o = 0$.

Noting that $f(s) = f(o) + \int_o^s f'(r)dr$, we have then

$$u(t) = \int_o^t e^{-(t-s)A} f(o)ds + \int_o^t dr \int_r^t e^{-(t-s)A} f'(r)ds .$$

But (see Lemma 1. 12 below)

$$A\int_o^t e^{-(t-s)A} ds = A\int_o^t e^{-sA} ds = 1 - e^{-tA} ,$$

$$A\int_r^t e^{-(t-s)A} ds = A\int_o^{t-r} e^{-sA} ds = 1 - e^{-(t-r)A} .$$

Hence $Au(t)$ exists and

$$Au(t) = (1 - e^{-tA})f(o) + \int_o^t (1 - e^{-(t-r)A})f'(r)dr$$
$$= f(t) - e^{-tA} f(o) - \int_o^t e^{-sA} f'(t-s)ds .$$

On the other hand

---

(1) This theorem is due to Phillips [8] .

$$\frac{d}{dt} u(t) = \frac{d}{dt} \int_0^t e^{-sA} f(t-s)ds = e^{-tA} f(o) + \int_0^t e^{-sA} f'(t-s)ds$$

Hence $\frac{d}{dt} u(t) = - Au(t) + f(t)$, as we wished to show.

By the way, the continuity of $\frac{d}{dt} u(t)$ and of $Au(t)$ are obvious from the above expressions.

Lemma 1.12. Let $-A \in (Bo)$. Then

$$A \int_r^t e^{-sA} ds = e^{-rA} - e^{-sA}, \qquad 0 \leq r < t.$$

Proof. If $u \in D_A$, we have $Ae^{-sA}u = - \frac{d}{ds} e^{-sA}u = e^{-sA}Au$ (which is continuous in t).

Hence

$$A \int_r^t e^{-sA} u \, ds = \int_r^t Ae^{-sA} u \, ds = (e^{-rA} - e^{-tA})u.$$

(The first equality is a direct consequence of the closure of A). For any $v \in X$, let $u_n \in D_A$, $u_n \to v$. Then $\int_r^t e^{-sA} u_n ds \to \int_r^t e^{-sA} v \, ds$ and $A(\int_r^t e^{-sA} u_n ds) = (e^{-rA} - e^{-tA}) u_n \to (e^{-rA} - e^{-tA}) v$. It follows, again by the closure of A, that $A\int_r^t e^{-sA} v \, ds$ exists and equals $(e^{-rA} - e^{-tA}) v$.

q. e. d.

Theorem 1.13. If $-A \in (H0)$, the continuous differentiability of $f(t)$ in the second part of Theorem 1.11 can be replaced by a Hölder continuity. Furthermore, $u(t)$ of (S_o) is analytic if $f(t)$ is analytic on $[0, T]$.

Proof. Again we may assume $u_o = 0$. Then

$$u(t) = \int_0^t e^{-(t-s)A}(f(s) - f(t))ds + \int_0^t e^{-(t-s)A} f(t)dt.$$

Therefore (see Lemma 1.12)

$$Au(t) = \int_0^t A e^{-(t-s)A}(f(s) - f(t))ds + (1 - e^{-tA}) f(t).$$

Note that the integral exists because $\left| Ae^{-(t-s)A} \right| \leq \frac{\text{const}}{t-s}$ (see Theorem 1. 7) and $|f(s) - f(t)| \leq \text{const}(t-s)^{\vartheta}$, $\vartheta > 0$.

This shows that $Au(t)$ exists and (closure of $A(t)$!)

$$A(t)u = A \int_0^t e^{-(t-s)A}(f(s) - f(t))\, ds + (1 - e^{-tA})f(t) .$$

On the other hand, the construction of $\partial u(t)/\partial t$ requires a little detour. We define

$$u_{\varepsilon}(t) = \int_0^{t-\varepsilon} e^{-(t-s)A} f(s)\, ds$$

Obviously $u_{\varepsilon}(t) \longrightarrow u(t)$ for $\varepsilon \longrightarrow 0$, locally uniformly in t . Also

$$\frac{d}{dt}\, u_{\varepsilon}(t) = e^{-\varepsilon A} f(t-\varepsilon) - \int_0^{t-\varepsilon} A\, e^{-(t-s)A} f(s)\, ds \ ;$$

the integral exists since $Ae^{-(t-s)A}$ is continuous for $s \leq t - \varepsilon$. But it is easy to see that the limit for $\varepsilon \to 0$ of this integral exists and equals $Au(t)$ (use again the Hölder continuity of $f(t)$), so that

$$\frac{d}{dt}\, u_{\varepsilon}(t) \to f(t) - Au(t) .$$

Moreover, this convergence is locally uniform in t . Hence it follows that $\frac{d}{dt}u(t)$ exists and equals $f(t) - Au(t)$, by a well known theorem in differential calculus.

If $f(t)$ is analytic, $u_{\varepsilon}(t)$ is also analytic : $du_{\varepsilon}(t)/dt$ given above exists for complex t in some neighborhood of the interval $[2\varepsilon , T]$. But $u_{\varepsilon}(t) \to u(t)$ is true locally uniformly in t for these complex t . It follows that $u(t)$ is analytic. q. e. d.

§ 2. THE CASE IN WHICH -A(t) ARE GENERATORS OF ANALYTIC SEMI-GROUPS WITH CONSTANT DOMAIN FOR $A(t)^h$.

1. First we note that the equation (E) is very simply dealt with if $A(t) \in B$ and strongly continuous in t. If we consider the homogeneous equation $du/dt + A(t)u = 0$, the solution can be constructed by a straightforward successive approximation :

$$u(t) = \sum_{k=0}^{\infty} u_k(t) \ , \quad u_0(t) = u_0 \ ,$$

$$u_k(t) = -\int_0^t A(s)u_{k-1}(s)\, ds \ , \qquad k = 1, 2, 3, \dots$$

This is equivalent to writing $u(t) = U(t,0)u_0$ and determining $U(t,0)$ from the differential equation $dU(t,0)/dt = -A(t)U(t,0)$, $U(0,0) = 1$, by successive approximation (the derivative is strong derivative). More generally, we can solve the differential equation

$$(1) \qquad \frac{\partial}{\partial t} U(t,s) = -A(t)U(t,s), \qquad U(s,s) = 1$$

by successive approximation. The family of operators $U(t,s)$ constructed in this way will be called the evolution operator (or the Green function).

The evolution operator has, in addition to (1), the following properties:

$$(2) \qquad \frac{\partial}{\partial s} U(t,s) = U(t,s)\, A(s)$$

$$(3) \qquad U(t,s)\, U(s,r) = U(t,r) \ .$$

To prove this, it is convenient to consider another differentiable equation

$$\frac{\partial}{\partial s} V(t,s) = V(t,s)\, A(s), \qquad V(t,t) = 1 \ .$$

This can again be solved by successive approximation.

Then

$$\frac{\partial}{\partial s} V(t,s)\, U(s,r) = V(t,s)(A(s) - A(s))\, U(s,r) = 0$$

so that $V(t,s)\,U(s,r)$ is independent of s. Putting $s = t$ and $s = r$, we obtain $U(t,r) = V(t,r)$, and hence $U(t,s)\,U(s,r) = U(t,r)$, q. e. d.

With the use of the evolution operator, the solution of (E) can be expressed by

$$\text{(S)} \qquad u(t) = U(t,0)u_0 + \int_0^t U(t,s)\, f(s)\, ds \; .$$

Now the above method does not work when $A(t)$ is not bounded. Therefore we want to construct the evolution operator for unbounded $A(t)$ by a limiting procedure, by approximating $A(t)$ by a sequence $A_n(t)$ of bounded operators (this is the way the semi-group e^{-tA} was constructed in Phillips' lectures as the limit of e^{-tA_n}, A_n being bounded). We choose

$$\text{(4)} \qquad A_n(t) = A(t)J_n(t) = n(1 - J_n(t)) \; , \quad J_n(t) = (1 + \frac{1}{n} A_n(t))^{-1} \; , \quad n = 1, 2, 3, ..$$

If $A_n(t)$ (or $J_n(t)$) is strongly continuous in t, we can construct the evolution operator $U_n(t,s)$ by the simple method described above. Then we want to show that s-lim $U_n(t,s)$ exists, which will be the evolution operator $U(t,s)$ for the unbounded case.

This method is seen to work under certain conditions on $A(t)$. We have namely

<u>Theorem 2. 1.</u>[1)] Assume that

i) $-A(t) \in$ (H0) uniformly for $0 \leq t \leq T$. This means that there are constants $M > 0$, $\omega > 0$ such that

---

(1) This theorem is first proved by Tanabe (see [10, 11, 12]). The last proposition regarding the analyticity is due to Komatsu [7]. The proof given here is somewhat different from theirs. See also Yosida [17].

$$|(\lambda + A(t))^{-1}| \leq \frac{M}{|\lambda|} \quad \text{for} \quad |\arg\lambda| \leq \frac{\pi}{2} + \omega$$

and

$$|A(t)^{-1}| \leq M .$$

ii) $D_{A(t)} = D$ is independent of t. This implies that $A(t)A(s)^{-1} \in B$ for any s and t.

iii) $A(t)\,A(0)^{-1}$ is Hölder continuous (in norm). This is equivalent to that

$$|A(t)\,A(s)^{-1} - 1| \leq M(t-s)^{\vartheta}, \qquad \vartheta > 0 .$$

Then there exist a unique evolution operator $U(t,s)$ with the following properties :

1) $U(t,s) \in B$ and strongly continuous for $0 \leq s \leq t \leq T$.

2) $U(t,s)\,U(s,r) = U(t,r)$, $U(t,t) = 1$.

3) $U(t,s)$ is strongly continuously differentiable in t for $t > s$, with

$$\frac{\partial}{\partial t} U(t,s) = -A(t)\,U(t,s) , \qquad |A(t)\,U(t,s)| \leq \frac{\text{const.}}{t-s} .$$

4) If $u \in D$, then

$$\frac{\partial}{\partial s} U(t,s)\,u = U(t,s)\,A(s)\,u , \qquad s \leq t .$$

If, in addition, $A(t)^{-1}$ is analytic in t, then $U(t,s)$ has an analytic continuation for complex s,t such that $|\arg(t-s)|$ are sufficiently small.

Remark 2. 2. a) ii) and iii) express that $A(t)$ depends on t "smoothly". Note that ω of ii) is equal to the $\omega - \xi$ of Def. 1. 5. for some fixed ξ .
b) The assumption $|A(t)^{-1}| \leq M$ is made only for simplicity; if this is not the case, then we may make a transformation $u(t) \longrightarrow e^{-\xi t} u(t)$ in (E) so that the new equation has $A(t)$ satisfying the above condition (this is allowed since we are interested only in a finite interval $0 \leq t \leq T$). c) The assumptions ii) and iii) will be weakened later. But it should be remarked that these

are satisfied if $A(t)$ is an operator in $X = L^2(\Omega)$ determined from an elliptic differential operator with smooth coefficients on a bounded region Ω with a smooth boundary, with the Dirichlet boundary condition. In this case $D_A(t) = H^{2m}(\Omega) \cap H_0^m(\Omega)$ and is independent of t.

2. Proof of Theorem 2. 1.

We construct the approximating operators $A_n(t)$ by (4). $A_n(t)$ is bounded by $|A_n(t)| \leq (M+1)n$ since $|J_n(t)| \leq M$. But the important fact is that $-A_n(t)$ belongs to the class (H0) uniformly in t and n in the sense stated in i) of Theorem 2. 1'. We have namely

(5) $$A_n(t)^{-1} = A(t)^{-1} + n^{-1}, \qquad |A_n(t)^{-1}| \leq M + n^{-1} \leq M + 1.$$

Also a straightforward computation leads to

(6) $$(\lambda + A_n(t))^{-1} = \frac{n^2}{(n+\lambda)^2}\left(\frac{n\lambda}{n+\lambda} + A(t)\right)^{-1} + \frac{1}{n+\lambda}$$

from which an elementary geometric consideration gives

(7) $$|(\lambda + A_n(t))^{-1}| \leq \frac{M'}{|\lambda|} \qquad |\arg\lambda| \leq \frac{\pi}{2} + \omega$$

where M' is in general different from M but may be taken independent of n. Finally, we have

(7') $$A_n(t)\,A_n(s)^{-1} = A_n(t)\,(A(s)^{-1} + \frac{1}{n}) = J_n(t)\,A(t)\,A(s)^{-1} + 1 - J_n(t),$$
$$A_n(t)\,A_n(s)^{-1} - 1 = J_n(t)\,(A(t)A(s)^{-1} - 1),$$

hence

(8) $$|A_n(t)\,A_n(s)^{-1} - 1| \leq M\,|A(t)A(s)^{-1} - 1| \leq M^2(t-s)^{\vartheta}$$

In particular $|A_n(t) - A_n(s)| \leq |A_n(t)\,A_n(s)^{-1} - 1|\,|A_n(s)| \leq$ $\leq M^2(M+1)\,n(t-s)^{\varrho}$, so that $A_n(t)$ is itself Hölder continuous.

Therefore, the evolution operator $U_n(t,s)$ for $A_n(t)$ can be constructed as stated above.

To prove that $s\text{-}\lim_{n\to\infty} U_n(t,s)$ exists, however, we have to deduce other expressions and estimates for $U_n(t,s)$. To this end we first note the identity

$$\frac{\partial}{\partial s} U_n(t,s)e^{-(s-r)A_n(r)} = U_n(t,s)(A_n(s) - A_n(r))e^{-(s-r)A_n(r)}$$

Integrating on $r \leq s \leq t$, one obtains

(9) $$U_n(b,r) = e^{-(t-r)A_n(r)} + \int_r^t U_n(t,s)\, K_n(s,r)ds\ ,$$

where

(10) $$K_n(s,r) = -\left[A_n(s)\, A_n(r)^{-1} - 1\right]\ A_n(r)e^{-(s-r)A_n(r)}\ .$$

$K_n(s,r)$ has the estimate

(11) $$\left| K_n(s,r)\right| \leq \frac{C}{(s-r)^{1-\vartheta}}\ ,$$

where C is a constant independent of n. This follows from (8) and $\left| A_n(r)e^{-(s-r)A_n(r)}\right| \leq C(s-r)^{-1}$, which is in turn a consequence of $-A_n(r) \in (H0)$ (Theorem 1. 7.).

(9) may be considered an integral equation for $U_n(t,r)$. This may be written symoblically

(12) $$U_n = U_n^{(o)} + U_n * K_n$$

and can be solved by successive approximation :

(13) $$U_n = \sum_{k=o}^{\infty} U_n^{(k)}\ , \qquad U_n^{(k)} = U_n^{(k-1)} * K_n\ .$$

The possibility of this successive approximation is guaranteed by the estimate (11) [1] for K_n and the estimate $\left|U^{(o)}(t,r)\right| \leq C$ (we use the same symbol

(1) The important fact is that K_n are dominated by an integrable kernel independent of n.

C to denote different constants). It follows that the series (13) is convergent uniformly, being majorized by a series independent of n .

Now we can let $n \to \infty$. Then

(14) $$U_n^{(o)}(t,s) \underset{s}{\to} U^{(o)}(t,s) = e^{-(t-s)A(s)} .$$

(14) is exactly the construction of the semi-group $e^{-\tau A(s)}$ as the limit of $e^{-\tau A_n(s)}$ (see Phillips' lectures).

Also

(15) $$K_n(t,s) \underset{s}{\to} K(t,s) = -(A(t)A(s)^{-1}-1)A(s)e^{-(t-s)A(s)}$$

by (7') and $J_n(t) \to 1$ (which is a basic fact proved in the generation of semi-groups, see Phillips) and

(16) $$A_n(s)e^{-\tau A_n(s)} \underset{s}{\to} A(s)e^{-\tau A(s)} , \qquad \tau > 0 ,$$

which follows from (3) of § 1. We note that the strong convergence (14) and (15) are uniform in s and t (for $t-s \geq \alpha > 0$ in (15)).

It follows from (14) and (15) that $U_n^{(1)}(t,s) = U_n^{(o)} * K_n(t,s) \underset{s}{\to} U^{(1)}(t,s)$, then $U_n^{(2)}(t,s) = U_n^{(1)} * K_n(t,s) \underset{s}{\to} U^{(2)}(t,s)$, and so on. In view of the fact that the series (13) is uniformly majorized, we conclude that

(17) $$U_n(t,s) = \sum_k U_n^{(k)}(t,s) \underset{s}{\to} U^{(k)}(t,s) \equiv U(t,s), \quad \left| U(t,s) \right| \leq C .$$

Since it is easily seen that the strong convergence (17) is uniform in s, t for $s \leq t$, U(t,s) is strongly continuous for $s \leq t$. Also 2) of Theorem 2.1. follows from the corresponding relation for $U_n(t,s)$.

3. Proof of Theorem 2. 1., continued.

We use another identity

$$\frac{d}{ds}\, e^{-(t-s)A_n(t)}\, U_n(s,r) = e^{-(t-s)A_n(t)}(A_n(t) - A_n(s))\, U_n(s,r) \;,$$

whence we obtain

$$U_n(t,r) = e^{-(t-s)A_n(t)} + \int_r^t e^{-(t-s)A_n(t)}(A_n(t)\,A_n(s)^{-1} - 1)A_n(s)\,U_n(s,r)ds.$$

Multiply this equation with $A_n(t)$ from left and write $Y_n(t,s) = A_n(t)\,U_n(t,s)$; then

(18) $$Y_n(t,r) = Y_n^{(o)}(t,r) + \int_r^t H_n(t,s)\; Y_n(s,r)ds$$

where

(19) $$\begin{cases} Y_n^{(o)}(t,s) = A_n(t)e^{-(t-s)A_n(t)} \\ H_n(t,s) = A_n(t)e^{-(t-s)A_n(t)}(A_n(t)A_n(s)^{-1} - 1). \end{cases}$$

(18) may be written symbolically as

(20) $$Y_n = Y_n^{(o)} + H_n \ast Y_n \;.$$

We want to solve (20) again by successive approximation :

(21) $$Y_n = \sum_{k=o}^{\infty} Y_n^{(k)} \;, \qquad Y_n^{(k)} = H_n \ast Y_n^{(k-1)} \;.$$

Here, however, we have a slight difficulty that did not exist in (13), for $Y_n^{(o)}$ has the uniform (independent of n) estimate $|\, Y_n^{(o)}(t,s) \,| \le C(t-s)^{-1}$ where $(t-s)^{-1}$ is <u>not</u> integrable. Thus the uniform estimate of $Y_n^{(1)}$ is not quite simple, although its existence is obvious ($A_n(t) \in B$!).

Here we give only the result :

(22) $$|\, Y_n^{(1)}(t,s) \,| \le \frac{C}{(t-s)^{1-\theta}}$$

of which the proof will be given in n. 7.

Once (22) is established, the further successive approximation proceeds smoothly, for the right member of (22) is integrable as well as that of

(23) $$\left| H_n(t,s) \right| \leq \frac{C}{(t-s)^{1-\theta}}$$

which is proved as in (11). By an argument similar to that given in n. 2, it follows that (21) is uniformly majorized and that each term $Y_n^{(k)}$ has a strong limit $Y^{(k)}$ for $n \to \infty$. Hence

(24) $$A_n(t)\, U_n(t,s) = Y_n(t,s) \xrightarrow[s]{} Y(t,s), \qquad \left| Y(t,s) \right| \leq \frac{C}{t-s}$$

(24) gives (see also (5))

$$U_n(t,s) = A_n(t)^{-1}\, Y_n(t,s) \xrightarrow[s]{} A(t)^{-1}\, Y(t,s)\,.$$

Since $U_n(t,s) \xrightarrow[s]{} U(t,s)$ by (17), we must have $U(t,s) = A(t)^{-1}Y(t,s)$. This means that $A(t)U(t,s)$ exists and equals $Y(t,s) \in B$ if $t > s$. Thus we have proved

(25) $$\left| A(t)\, U(t,s) \right| \leq \frac{C}{t-s}$$

The differentiability of $U(t,s)$ is proved in the following way. Since $\partial U_n(t,s)u/\partial t = -A_n(t)U_n(t,s)u$ and $U_n(t,s)u \to U(t,s)u$, $A_n(t)U_n(t,s)u \to$ $\to Y(t,s)u = A(t)U(t,s)u$ uniformly for $t \geq s+a$, it follows that $\partial U(t,s)u/\partial t$ exists and is equal to $-A(t)U(t,s)u$. That is, $U(t,s)$ is strongly differentiable in t for $t > s$, with the strong derivative $-A(t)U(t,s) \in B$.

Similarly, we have

$$\frac{\partial}{\partial s}\, U_n(t,s)u = U_n(t,s)A_n(s)u \quad .$$

If $u \in D = D_{A(s)}$, we have $A_n(s)u \to A(s)u$, $n \to \infty$, uniformly in s (see Phillips) so that $\frac{\partial}{\partial s}\, U_n(t,s)u \to U(t,s)A(s)u$. The same argument as above then proves that $\frac{\partial}{\partial s} U(t,s)u = U(t,s)A(s)u$.

If $A(t)^{-1}$ is analytic in t in a neighborhood Δ of $0 \leq t \leq T$, the same is true with $A_n(t) = (A(t)^{-1} + n^{-1})^{-1}$. Therefore $U_n(t,s)$ can be continued analytically to $t \in \Delta$, $s \in \Delta$. Now the expression of $U_n(t,s)$ by the series (13) holds true when the variables t, s, r are supposed to lie on a straight line in Δ having a small angle ϑ with the positive real axis, uniformly with respect to ϑ , and each term $U_n^{(k)}$ is seen to converge for $n \to \infty$ to $U^{(k)}$ uniformly (on the line as well as in ϑ). Thus $U_n(t,s)$ converges strongly and (locally) uniformly to a $U(t,s)$ as long as $|\arg(t-s)|$ are sufficiently small. It follows that $U(t,s)$ is <u>strongly</u> analytic in such a region of t and s . But since strongly analyticity is equivalent to analyticity (in norm), $U(t,s)$ is analytic. This completes the proof of Theorem 2. 1.

4. We now consider the inhomogeneous equation (E).

<u>Theorem 2. 3.</u> Let the assumptions of Theorem 2. 1. be satisfied. Then the conclusions of Theorem 1. 13. are true (with (E_0) and (S_0) replaced by (E) and (S), respectively).

<u>Proof.</u> Almost the same as for Theorem 1. 13. The only modification required is to note that

$$A(t)U(t,s) = A(t)e^{-(t-s)A(t)} + Y'(t,s)$$

$$Y'(t,s) = \sum_{k=1}^{\infty} Y^{(k)}(t,s) , \qquad |Y'(t,s)| \leq \frac{C}{(t-s)^{1-\vartheta}}$$

see (20), (22), (24). Hence

$$A(t)\int_r^t U(t,s)ds = \int_r^t A(t)U(t,s)ds = 1 - e^{-(t-r)A(t)} + \int_r^t Y'(t,s)ds$$

by Lemma 1. 12 ($Y'(t,s)$ is absolutely integrable).

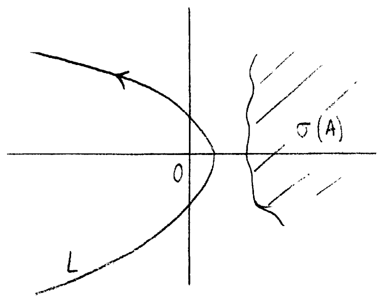
5. Generalizations.

To improve Theorems 2. 1. and 2. 3. , we need the fractional powers $A(t)^{\alpha}$ of $A(t)$.

When $-A \in (B0)$, the fractional powers A^{α} can be defined in a natural way [1]. Here we assume, for simplicity, that $A^{-1} \in B$ in addition. Then we can first construct the Dunford integral

$$(26) \qquad A^{-\alpha} = -\frac{1}{2\pi i} \int_L z^{-\alpha} (z - A)^{-1} dz \in B , \qquad \alpha > 0 .$$

where L is a curve from $-\infty$ to $-\infty$ passing between $z = 0$ and $\sigma(A)$. The integral converges absolutely since $|(z-A)^{-1}| \leq M/\mathrm{Im}(-z)$.
Since this is a Dunford integral it is easy to see that $A^{\alpha+\beta} = A^{\alpha} A^{\beta}$.
If $0 < \alpha < 1$, L may be taken as the double ray $(0, -\infty)$, yielding



$$(27) \qquad A^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^{\infty} \lambda^{-\alpha} (\lambda + A)^{-1} d\lambda .$$

Then we define A^{α} as the inverse of $A^{-\alpha}$; note that $A^{-\alpha}$ is invertible since $A^{-\alpha} u = 0$ implies $A^{-n} u = A^{-(n-\alpha)} A^{-\alpha} u = A^{-n} u = 0$, $u = 0$, where n is a positive integer larger than α .

We need also the following expression, which is valid for $-A \in (H0)$.

$$(27') \qquad A^{\alpha} e^{-\tau A} = \frac{1}{2\pi i} \int_C (-\lambda)^{\alpha} e^{\lambda\tau} (\lambda + A)^{-1} d\lambda .$$

This can be proved by verifying that (27') gives $e^{-\tau A}$ when multiplied by (26) (cf. the proof of Theorem 1. 7). It follows from (27') that

$$(27'') \qquad | A^{\alpha} e^{-\tau\alpha} | \leq \frac{C}{|\tau|^{\alpha}}$$

(1) These fractional powers are considered by many authors; see, for example, [3] , [4] , [16] and the references given there.

We can now state generalization of Theorems 2. 1. and 2. 3.

Theorem 2. 4. [1] Assume that

i) $-A(t) \in (H0)$ uniformly (as in i), Theorem 2. 1.).

ii) $D_{A(t)^h} = D_h$ = const. for some $h = 1/m$ with a positive integer m. This implies that $A(t)^h A(s)^{-h} \in B$ for any s and t.

iii) $A(t)^h A(s)^{-h}$ is Hölder continuous with an exponent $\rho > 1 - h$, so that $\| A(t)^h A(s)^{-h} - 1 \| \leq M(t-s)^{\rho}$.

Then the conclusions of Theorem 2. 1. are true (with D replaced by D_h in 4)). In the last statement of Theorem 2. 1. (analyticity), the analyticity of $A(t)^{-h}$ should be assumed.

Theorem 2. 5. Under the assumptions of Theorem 2. 4., the conclusions of Theorem 2. 3. are true.

Remark 2. 6. Theorem 2. 1. is a special case of Theorem 2. 4. for $m = 1$. The assumption that $D_{A(t)^h}$ = const. is supposed to be weaker than that $D_{A(t)}$ = const., but there is no general proof valid for Banach spaces X. In any case this is true for accretive operators $A(t)$ in a Hilbert space. In other words, $D_A = D_B$ implies $D_{A^\alpha} = D_{B^\alpha}$ if X is a Hilbert space and $-A$, $-B \in (B0)$ are such that $Re(Au, u) \geq 0$, $Re(Bu, u) \geq 0$ $(u \in D_A = D_B)$. Furthermore, it has been proved by Lions that, when A is an operator in $X = L^2(\Omega)$ determined from a strongly elliptic differential operator of order $2m$ on a domain Ω with a smooth boundary, A^α has a domain independent of the coefficients or of boundary conditions for $\alpha < 1/4m$.

Outline of the proof. Here it is impossible to give a complete proof which is in principle the same as the proof of Theorem 2. 1. We shall indicate briefly the essential points in the proof.

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(1) A similar theorem was stated by Sobolewski [9] in the special case where A(t) are a positive self-adjoint operator in a Hilbert space.