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Dario Graffi (Ed.)

Materials with Memory

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CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C.I.M.E.)

I Ciclo - Bressanone dal 2 all'11 giugno 1977

MATERIALS WITH MEMORY

Coordinatore: Prof. D. GRAFFI

CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C.I.M.E.)

PERIODIC PROBLEMS IN THERMOVISCOELASTICITY

R. BOUC e G. GEYMONAT

Corso tenuto a Bressanone dal 2 all'11 giugno 1977

Periodic Problems in Thermoviscoelasticity

Two Seminars given at the Centro Internazionale Matematico Estivo, Bressanone, Italy, june 3-11, 1977.

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We thank Professor D. GRAFFI for his kind invitation to this talk, the aim of which is to give a survey of some recent work done at Laboratoire de Mécanique et d'Acoustique of C.N.R.S. Marseille, partly in collaboration with M. JEAN, B. NAYROLES and M. RAOUS, specially during the second author's year of leave from the Politecnico di Torino.

Contents

Introduction

Following the fundamental work of V. Volterra |37|,|38| ,|39| ,|40| , hereditary phenomena in mechanics have been deeply studied.

A great part of the work that has been done (see|18 |,|35 |,|36 |,...) treats the case called, by Volterra himself, "the case of closed cycle" (see T. VOGEL|34 |) which corresponds to the case where the relaxation function in viscoelasticity is of the type $G(t-\tau)$.

However, in 1907, HATT|19| has discovered the phenomenon of creep in concrete which presents stress- independent deformations which, in addition to thermal dilatation, includes shrinkage ; the material properties of concrete change indeed as a result of internal chemical reactions and the deformation problem is coupled with complicate moisture diffusion through the material, as well as heat conduction. For these reasons, in a first approximation, concrete may be regarded as an aging viscoelastic material whose creep law can be written in a rate-type form, i.e. as a system of first-order differential equations, involving hidden strains, with time-dependent coefficients(x).

More recently it appears that also for other materials, especially polymers in a temperature depending situation, the relaxation function is not of type $G(t - \tau)$ but following a fundamental remark of Morland and Lee|27 |, the relaxation function can be written as $G(\xi - \xi')$ where $\xi = \xi(\theta)$ is the reduced time (see also PIPKIN |31|).

From another point of view the extension of phenomenological laws based on spring and dashpot models to the temperature depending case has been proposed by many experimentalists (see e.g. | 4 |) specially for metals.

In this paper we shall recall in §1 some results on the continuum mechanics of materials with hidden coordinates (indeed hidden strains) and some consequences of the Clausius-Duhem inequality on the constitutive equations due to Coleman-Gurtin|12 | and Bowen| 10 |.

(x)For a very deep review of the basic facts on this subject see Z.P. BAŽANT| 3 |.

Because we are interested in the study of a phenomenon with high temperature variations, we develop in §2, in the case of infinitesimal strains, a linearization of the equations obtained in §1 only with respect to strains and hidden coordinates. We conclude this analysis in §3, with some remarks on the possibility of uncoupling the nonlinear heat equation, similar to those developped by Crochet-Naghdi | 13 | for thermorheologically simple solids.

In §4 we recall very briefly how the nonlinear heat equation obtained in this way can be studied in the framework of nonlinear evolution equations as developped in the book of Lions | 24 |.

In §5 we start the study of the equation of motion (with temperature as a data, i.e. a given function of time and space-variables), recalling some results on duality and virtual work principle. In §6 we consider a constitutive equation of Maxwell-type where the "stiffness" and "viscosity" matrix are temperature-dependent and thus are time dependent. More precisely the temperature is T-periodic in time and therefore the stiffness and the viscosity are also T-periodic. With this constitutive equation we survey, in §7 and §8 some results concerning existence, uniqueness, asymptotic stability of a T-periodic stress-field for the dynamic and quasi-static periodic bilateral problem (| 6 |, | 9 |, | 17 |) and also for the quasi-static Signorini unilateral periodic problem (| 7 |, | 8 |). We refer to the lectures of G. Fichera in the present session for the corresponding Cauchy-problems.

The applications of our results on the thermal fatigue of metals due to cycle heating and cooling will be developped in the thesis of M. Raous | 32 |. By lack of time we cannot develop these first results ; we can only say that the numerical experiments agree with the experimental results of F.K.G. ODQVIST and N.G. OHLSON | 30 | "The virginal specimens behaved in a normal way, whereas those already cracked apparently proved to be stronger against the formation of new cracks".

1- Background on the mechanics of continuous medium with hidden variables.

1.1- The mechanical and thermal behaviour in the time interval $\mathcal{C} \subset \mathbb{R}$, of a nonpolar body occupying the reference configuration $\Omega \subset \mathbb{R}^3$ at time $t_0 \in \mathcal{C}$ is completely determined by a vector function $p(X,t)$ (giving the position at time t of a material point which has the position X in the reference configuration Ω) and by a positive scalar function $\theta(X,t)$ (giving the absolute temperature at time t of a material point which has the position X in the reference configuration Ω).

As usual we define $F(X,t) = \text{Grad } p(X,t)$, the deformation gradient tensor and we shall assume that $p(X,t)$ is always smoothly invertible, i.e.

(1.1) $$\det F(X,t) > 0 \qquad \text{for all} \quad t \in \mathcal{C}$$

Using Lagrange's coordinates, the local form of the laws of balance of linear momentum, of moment of momentum and of energy are the following(x) (xx) (see |16|) :

(1.2) $$\text{Div } FS + \rho_0 f = \rho_0 \ddot{p}$$

(1.3) $$S = S^*$$

(1.4) $$\rho_0 \dot{\varepsilon} = \text{tr } S\dot{E} - \text{Div } q + \rho_0 r$$

where $\rho_0 = \rho_0(X)$ is the mass density in the reference position, S is the symmetric second Piola-Kirchoff (or Lagrangean) stress tensor, $E = \frac{1}{2}(F^*F - \mathbb{1})$ is the Lagrangean strain tensor, $\dot{E} = \frac{\partial E}{\partial t}$ is the Lagrangean strain rate, $\rho_0 \ddot{p} = \rho_0 \dfrac{\partial^2 p}{\partial t^2}$ is the inertia force, f is the body force field per unit

---

(x) If A is an $m \times n$ matrix, A^* denotes the transposed matrix.
(xx) $\text{tr}(.)$ = trace of (.).

mass, ε is the internal energy of the body per unite mass, q is the heat conduction vector, r is the heat supply field per unit mass.

Let us also recall that the law of the conservation of the mass allows us to compute the mass density at the time t with the formula

$$\det F = \frac{\rho_0}{\rho}$$

The local Clausius-Duhem inequality

$$(1.5) \qquad \rho_o \dot{\eta} + \mathrm{Div}\left(\frac{q}{\theta}\right) - \rho_o \frac{r}{\theta} \geqslant 0$$

where η is the specific entropy per unit mass can also be written, using (1.4), in the form

$$(1.6) \qquad -\rho_o \dot{\varepsilon} + \rho_0 \theta \dot{\eta} + \mathrm{tr}(S \mathring{E}) - \frac{q^* g}{\theta} \geqslant 0$$

where $g = \mathrm{Grad}\,\theta$

Defining the Helmoltz free energy per unit mass by

$$(1.7) \qquad \psi = \varepsilon - \theta \eta$$

we can also write (1.6) in the form

$$(1.8) \qquad -\rho_0 \dot{\psi} - \rho_o \eta \dot{\theta} + \mathrm{tr}(S \dot{E}) - \frac{q^* g}{\theta} \geqslant 0$$

1.2- The characteristics of material composing the body are usually stated by additional equations, the so-called constitutive equations, which give the stress, the internal energy, the entropy and the heat conduction in terms of the Lagrangean strain tensor and the temperature field. Obviously the constitutive equations depend on the properties of the material that we are modelling, and in the following we construct a model for solid-like materials (e.g. metals, polymers, concrete,...) whose response depends to a large extent on their past history (a qualitative explanation of this fact can be given in terms of various microstructural rearrangements due to dislocations motions, longchain molecules, phase transformations,...).

We will account for such microscopic structural rearrangements by the introduction of additional state variables called internal or hidden coordinates and denoted collectively by ξ which in a certain average global sense represents the internal changes.

As is pointed out by S. Nemat-Nasser (|29 |p. 110) :

"The representation is *macroscopic* in the sense that there may exist multiple (in fact, probably infinitely many) microstates corresponding to the same values of these coordinates. However, inasmuch as these coordinates are characterized by certain constitutive relations involving various parameters, which are fixed by means of suitable macroscopic experiments, they signify the most phenomenologically dominant aspects of the microstructural changes".

One can assume that the hidden coordinates are various tensorial quantities that transform in a suitable way under a change of frame, here we shall assume for simplicity that ξ is a symmetric positive definite tensor invariant by orthogonal change of frame.

1.3- A thermodynamic process is a set of functions of $X \in \Omega$ and $t \in \mathcal{C}$

$$\Lambda = \{p, \theta, S, f, \varepsilon, q, r, \eta\}$$

that satisfy (1.2), (1.3) and (1.4).

In order to be frame indifferent, the lagrangean stress, the free energy, the entropy and the heat flux are defined as functions of the material point and of the actual values of the state variables $E, \dot{E}, \theta, g, \xi$ (the thermodynamic state) :

$$(1.9) \qquad S = \hat{S}(X, E, \dot{E}, \theta, g, \xi)$$

$$(1.10) \qquad \psi = \hat{\psi}(X, E, \dot{E}, \theta, g, \xi)$$

$$(1.11) \qquad \eta = \hat{\eta}(X, E, \dot{E}, \theta, g, \xi)$$

$$(1.12) \qquad q = \hat{q}(X, E, \dot{E}, \theta, g, \xi)$$

In order to fix the variation of the hidden coordinate ξ we shall assume (x) :

<u>For all $X \in \Omega$, there exists a function h of $E, \dot{E}, \theta, g, \xi$ such that along any process during the time interval $\mathscr{C}$</u>

(1.13) $$\dot{\xi} = h(X, E, \dot{E}, \theta, g, \xi).$$

<u>Moreover for all $t_0 \in \mathscr{C}$ and all ξ_0 there exists a unique $\xi(X,t)$ satisfying (1.13) for all $t \in \mathscr{C}$ and $\xi(X,t_0) = \xi_0(X)$.</u>

1.4- The constitutive equations of materials that we have in mind are based on analogies to spring-and-dashpot models ; indeed these simple models display qualitatively retarded-elastic, creep and relaxation phenomena that are encountered in polymers, concrete, metals.

Example 1 - (Thermoelasticity)

We take $\xi_0 = h \equiv 0$ and $S = K(\theta)E + A(\theta)$. We have the usual thermo-elasticity. If θ_0 is the reference temperature in the reference configuration, without stress, we must write $A(\theta) = -K(\theta)\chi(\theta)(\theta - \theta_0)$ where $\chi(\theta)$ is the thermal dilatation tensor. $e_0 = -\chi(\theta)(\theta - \theta_0)$ is the thermal dilatation and thus $S = K(\theta)(E + e_0)$.

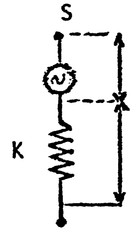
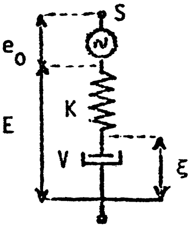


Fig.1

Example 2- (Maxwell body)



We have the relations

(1.14) $$\begin{cases} S = K(\theta)\,[E + e_0 - \xi] \\ S = V(\theta)\,\dot{\xi} \end{cases}$$

from which it follows

(1.15) $$\dot{\xi} = V^{-1}(\theta)\,K(\theta)\,(E + e_0 - \xi)$$

Fig.2

(x) As has been pointed out by G. CAPRIZ and L.M. SAHA |11|(see also F. SIDOROFF |33|) the Clausius-Duhem inequality implies that either $\dot{\xi}$ depends on the other fields or $\hat{\psi}$ is independent of ξ.

One can also consider N Maxwell elements in parallel. This model is very interesting for concrete (see Z.P. BAŽANT | 3 |, where it is also studied a possible dependence from the temperature and the humidity).

Example 3- (Standard Solid)

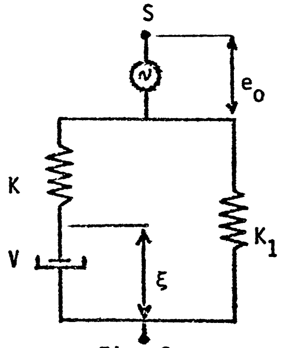


Fig. 3

$$S = K_1(\theta)[E + e_0] + K(\theta)[E + e_0 - \xi]$$

$$\dot{\xi} = V^{-1}(\theta) K(\theta) [E + e_0 - \xi]$$

Example 4- (Jeffrey's element)

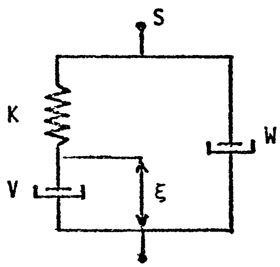


Fig. 4

$$S = W(\theta)\dot{E} + K(\theta)[E - \xi]$$

$$\dot{\xi} = V^{-1}(\theta) K(\theta) [E - \xi]$$

With respect to the choice of a good model and the influence of the thermodynamics we wish only quote S. NEMAT-NASSER (| 29 | p. 110) :
"In general, the selection of the hidden coordinates represents a significant problem. An experimentalist can only monitor certain "inputs" and measure certain "outputs". The material then represents a black box, whose internal structure is manifested through such input-output relations.
The optimal selection of suitable internal variables, minimum in number, which provides maximum information for a given input-output setup, is an interesting nontrivial problem outside the realm of thermodynamics.

Thermodynamics can only provides a general framework within which one must operate. The detailed selection of the parameters, however, must be guided by other considerations".

1.5- We shall now recall here some results essentially obtained by B.D. COLEMAN and M.E. GURTIN | 12 | and by R.M. BOWEN | 10 | on the thermodynamics with hidden variables.

It is clear that in order to specify a process for the body all we need do is to give the motion $p(X,t)$, the temperature field $\theta(X,t)$ and the value $\xi_0(X)$ of the hidden variables at some instant $t_0 \in \mathcal{G}$, for then all the other quantities can be computed. Moreover from the conservation laws one can choose the fields $f(X,t)$ and $r(X,t)$ to maintain the motion and the temperature.

Theorem 1 (| 12 |,| 10 |).

i) The Clausius-Duhem inequality (1.8) is verified for all $X \in \Omega$ and $t \in \mathcal{G}$ and all admissible thermodynamic process Λ (i.e. a thermodynamic process which is consistent with (1.9) to (1.13)) if and only if the constitutive equations (1.9),...,(1.13) satisfy :

(1.18) $\hat{\psi}$ and $\hat{\eta}$ are independent from $\dot{E}$ and g

(1.19) $$\eta = -\frac{\partial\hat{\psi}}{\partial\theta}$$

(1.20) $$\mathrm{tr}(\hat{S} - \rho_0 \frac{\partial\hat{\psi}}{\partial E})\dot{E} - \mathrm{tr}\,\rho_0 \frac{\partial\hat{\psi}}{\partial\xi}\dot{\xi} - \frac{\hat{q}^{x} g}{\theta} \geqslant 0$$

ii) If we assume *a priori* that h, $\hat{S}$ and $\hat{q}$ are independent from $\dot{E}$, the Clausius-Duhem inequality (1.8) is verified if and only if

(1.18 bis) $\hat{\psi}$ and $\hat{\eta}$ are independent from $\dot{E}$ and g

(1.19 bis) $$\eta = -\frac{\partial\hat{\psi}}{\partial\theta} \quad \text{and} \quad S = \rho_0 \frac{\partial\hat{\psi}}{\partial E}$$

(1.20 bis) $$\mathrm{tr}\,\rho_0 \frac{\partial\hat{\psi}}{\partial\xi}\dot{\xi} + \frac{\hat{q}^{x} g}{\theta} \leqslant 0$$

1.6- The results obtained in theorem 1 imply some simplifications in the energy equation (1.4) ; indeed in the case i) such equation can be written

(1.4 bis) $$\rho_0 \theta \dot{\eta} + \mathrm{tr}\, \rho_0 \frac{\partial \hat{\psi}}{\partial \xi} \dot{\xi} = \mathrm{tr}(S - \rho_0 \frac{\partial \hat{\psi}}{\partial E})\dot{E} - \mathrm{Div}\, \hat{q} + \rho_0 r$$

and in the case ii) such equation can be written

(1.4 ter) $$\rho_0 \theta \dot{\eta} + \mathrm{tr}\, \rho_0 \frac{\partial \hat{\psi}}{\partial \xi} \dot{\xi} = - \mathrm{Div}\, \hat{q} + \rho_0 r$$

where $\eta = - \frac{\partial \hat{\psi}}{\partial \theta}$.

1.7- As simple choice of the constitutive equations (1.9)-(1.13), suggested by the examples 1,2, 3,4 and compatible with the Clausius-Duhem inequality (see th.1) we shall assume from now on the following :

(1.21) $$\hat{\psi}(X,E,\theta,\xi) = A_0(X,\theta) + A_1(X,\theta)\xi + A_2(X,\theta)E + \frac{1}{2}\mathrm{tr}\, A_3(X,\theta)EE + \mathrm{tr}\, A_4(X,\theta)E\xi + \frac{1}{2}\mathrm{tr}\, A_5(X,\theta)\xi\xi$$

(1.22) $$\hat{S}(X,E,\dot{E},\theta,g,\xi) = B_0(X,\theta,g)\dot{E} + \rho_0 A_2(X,\theta) + \rho_0 A_3(X,\theta)E + \rho_0 A_4(X,\theta)\xi$$

(1.23) $$\hat{q}(X,E,\dot{E},\theta,g,\xi) = - k(X,\theta)g$$

(1.24) $$h(X,E,\dot{E},\theta,g,\xi) = - B_1(X,\theta,g)\,[A_1(X,\theta) + A_4(X,\theta)E + A_5(X,0)\xi]$$

where : B_0, k and B_1 are positive semi-definite tensors in order to ensure the validity of (1.20) ; moreover B_0, A_2, A_3 and A_4 are tensorial quantities symmetric in the first 2 indices in order to ensure the validity of (1.3), and B_1 is symmetric according to the Onsager principle.

Let us also remark that the expression of the entropy follows from (1.21) and (1.19).

2- A linearization result.

2.1- We shall now study what kind of simplification can be achieved in the equations obtained in § 1 in the hypothesis of the infinitesimal strains ; however we shall made no assumptions on the variation of the temperature θ (see M. J. CROCHET-P.M. NAGHDI |13| for analogous considerations in the case of thermorheologically simple solids).

More precisely, let us write the equations of § 1 in a non-dimensional set-up and let us define

$$(2.1) \qquad u_i(X,t) = p_i(X,t) - X_i \qquad i = 1, 2, 3 ;$$

$$\delta = \max \left\{ \sup_{i,X,t} \left| \frac{\partial u_i}{\partial X} \right| , \sup_{i,X,t} \left| \frac{\partial^2 u_i}{\partial X \, \partial t} \right| \right\}$$

In the sequel we assume that δ is small with respect to the unity.

We shall write that a function ϕ is $O(\delta^n)$ for $n \geq 0$ if there exists a constant $C > 0$ such that $|\phi| \leq C \delta^n$ uniformly in all the domain of definition of ϕ .

To construct the linearized system we shall only take the terms containing the lowest powers of δ .

2.2- From (2.1) we obtain

$$(2.2) \qquad F = \mathbb{1} + \text{Grad } u \qquad , \quad \dot{p} = \dot{u}$$

$$(2.3) \qquad \left\{ \begin{aligned} E &= \frac{1}{2} [\text{Grad } u + (\text{Grad } u)^{*}] + O(\delta^2) \\ \dot{E} &= \frac{1}{2} [\text{Grad } \dot{u} + (\text{Grad } \dot{u})^{*}] + O(\delta^2) \end{aligned} \right.$$

Therefore if we define

$$(2.4) \qquad \tilde{E} = \frac{1}{2} [\text{Grad } u + (\text{Grad } u)^{*}] \ , \ \dot{\tilde{E}} = \frac{1}{2} [\text{Grad } \dot{u} + (\text{Grad } \dot{u})^{*}]$$

then $\tilde{E} = O(\delta)$, $\dot{\tilde{E}} = O(\delta)$ and

(2.3 bis) $$E = \tilde{E} + O(\delta^2) \qquad \dot{E} = \dot{\tilde{E}} + O(\delta^2)$$

Moreover remarking that $\frac{1}{\det F} = 1 - \mathrm{Div}\, u + O(\delta^2)$ we find that the mass density at time t is given by

(2.5) $$\rho = \rho_0(1 - \mathrm{Div}\, u + O(\delta^2)) = \rho_0 + O(\delta)$$

and so we can consider, in a first approximation, that the mass density is time-independent, because $\rho_0 = O(1)$.

2.3- In order to linearize the equation of motion (1.2) we need some informations on the order of magnitude of the different terms that appear in (1.22). These informations are deduced from the following restrictions on the constitutive equations that will be better discussed on two examples.

Let us consider first the following initial value problem

$$\dot{y} + B_1(X, \theta, g)\, A_5(X, \theta)y = \phi(X, \theta, g) \quad ; \quad y(t_0) = 0$$

where B_1, A_5 are defined in (1.24) and where we assume $B_1 = O(1)$, $A_5 = O(1)$ and $\phi = O(\delta^n)$, $n \geq 0$. We have existence and uniqueness of the solution for all $t \in \mathscr{C}$ and we can write

$$y \cdot \frac{dy}{dt} + y \cdot B_1 A_5 y = y \cdot \phi$$

so that we deduce

$$\frac{1}{2} \frac{d|y|^2}{dt} \leq C_1 |y|^2 + |y|\,|\phi| \leq C_1 |y|^2 + \frac{1}{2}(|y|^2 + C_2\delta^{2n})$$

and by Gronwall Lemma

$|y(t)|^2 \leq C_3\, \delta^{2n} \quad \forall t \in \mathscr{C}$ provided $\mathscr{C}$ be bounded.
C_1, C_2, C_3 are positive absolute constants. We can prove now easily the following Lemma.

Lemma 1-

Let h be given by (1.24) and let us consider the following initial value problem

$$\dot{\xi} = h(X, E, \dot{E}, \theta, g, \xi) \quad ; \quad \xi(t_0) = \xi_0 \quad , \quad t_0 \in \mathscr{C} \tag{2.6}$$

Let $\mathscr{C}$ be bounded and

(H1) $B_1(X, \theta, g) = O(1)$, $A_5(X, \theta) = O(1)$, $A_4(X, \theta) = O(1)$

(H2) $A_1(X, \theta) + A_5(X, \theta)\ \xi_0(X) = O(\delta)$,

then we have

$$\begin{cases} \xi(t) = \tilde{\xi}(t) + O(\delta^2) \quad , \quad \tilde{\xi}(t) - \xi_0 = O(\delta) \\ \dot{\xi}(t) = \dot{\tilde{\xi}}(t) + O(\delta^2) \quad , \quad \dot{\tilde{\xi}}(t) = O(\delta) \end{cases} \tag{2.7}$$

where $\tilde{\xi}(t)$ is the unique solution of

$$\dot{\tilde{\xi}} = h(X, \tilde{E}, \dot{\tilde{E}}, \theta, g, \tilde{\xi}) \quad , \quad \tilde{\xi}(t_0) = \xi_0$$

Proof-

Take $y = \tilde{\xi} - \xi_0$ and $\phi = -B_1[A_1 + A_4\tilde{E} + A_5\xi_0] = O(\delta)$, then we have $y = O(\delta)$. Putting now $\xi - \tilde{\xi} = y$ and $\phi = -B_1A_4(E-\tilde{E}) = O(\delta^2)$ we have $y = O(\delta^2)$. Q.E.D.

Recalling (1.22),(2.3 bis) and (2.7) we can now write

$$\begin{aligned} \hat{S}(X,E,\dot{E},\theta,g,\xi) = {} & B_0(X,\theta,g)\dot{\tilde{E}} + \rho_0A_2(X,\theta) + \rho_0A_3(X,\theta)\tilde{E} + \\ & \rho_0A_4(X,\theta)(\tilde{\xi} - \xi_0) + \rho_0A_4(X,\theta)\xi_0 + \\ & B_0(X,\theta,g)(\dot{E} - \dot{\tilde{E}}) + \rho_0A_3(X,\theta)(E - \tilde{E}) + \\ & \rho_0A_4(X,\theta)(\xi - \tilde{\xi}). \end{aligned}$$

Let us denote respectively by θ_0 and S_0 the temperature field and the second Piola-Kirchoff stress tensor in the reference configuration (where $E = \dot{E} = 0$), we have

(2.8) $$S_o(X) = \rho_o \left[A_2(X,\theta_o) + A_4(X,\theta_o)\, \xi_o(X) \right]$$

from which we define $B_2(X, \theta, \theta_o)$ by

(2.9) $$\rho_o\left[A_2(X,\theta) + A_4(X,\theta)\, \xi_o(X)\right] = S_o(X) + \rho_o B_2(X,\theta,\theta_o)\,(\theta - \theta_o).$$

We shall also made the following assumptions

(H3) $$B_o(X,\theta,g) = O(1)\ , \quad \rho_o A_3(X,\theta) = O(1)\ , \quad \rho_o A_4(X,\theta) = O(1)$$

(H4) $$S_o(X) + \rho_o B_2(X,\theta,\theta_o)\,(\theta - \theta_o) = O(\delta).$$

Having done the good hypothesis we find that

$$\begin{aligned} \hat{S}(X, E, \dot{E}, \theta, g, \xi) = {} & B_o(X,\theta,g)\,\dot{\tilde{E}} + \rho_o A_3(X,\theta)\,\tilde{E} + \\ & \rho_o A_4(X,\theta)\,(\tilde{\xi} - \xi_o) + \rho_o B_2(X,\theta,\theta_o)\,(\theta - \theta_o) + \\ & S_o(X) + O(\delta^2) \end{aligned}$$

and so we can define

(2.10) $$\begin{aligned} \tilde{S} = {} & B_o(X,\theta,g)\,\dot{\tilde{E}} + \rho_o A_3(X,\theta)\,\tilde{E} + \rho_o A_4(X,\theta)\,(\tilde{\xi} - \xi_o) + \\ & \rho_o B_2(X,\theta,\theta_o)\,(\theta - \theta_o) \end{aligned}$$

and we obtain

(2.11) $$\begin{cases} S = \tilde{S} + S_o + O(\delta^2) \\ FS = \tilde{S} + S_o + O(\delta^2) \end{cases}$$

We can then take as <u>linearized equation of motion</u> the following

(2.12) $$\mathrm{Div}\left[\tilde{S} + S_o\right] + \rho_o f = \rho_o \ddot{u}$$

with from (2.9), (2.10)

$$\tilde{S} + S_o = B_o(X,\theta,g)\,\dot{\tilde{E}} + \rho_o A_3(X,\theta)\tilde{E} + \rho_o A_4(X,\theta)\tilde{\xi} + \rho_o A_2(X,\theta)$$

2.4- In order to linearize the energy equation (1.4 bis) we remark first of all that (H3) implies

$$(2.13)\qquad \mathrm{tr}(S - \rho_o \frac{\partial\hat{\psi}}{\partial E})\, \overset{\circ}{E} = \mathrm{tr}\, B_o(X,\theta\,,\,g)\, \overset{\circ}{E}\, \overset{\circ}{E} = O(\delta^2)$$

and the hypothesis (H2) implies

$$(2.14)\qquad \mathrm{tr}\,\rho_o \frac{\partial\hat{\psi}}{\partial \xi}\, h = \mathrm{tr}\,\rho_o \frac{\partial\hat{\psi}}{\partial \xi}\, \dot{\xi} = O(\delta^2)$$

Moreover we find from (1.21) and $\eta = -\dfrac{\partial\hat{\psi}}{\partial\theta}$,

$$-\overset{\circ}{\eta} = \Big[\frac{\partial^2 A_o}{\partial\theta^2} + \frac{\partial^2 A_1}{\partial\theta^2}\,\xi + \frac{\partial^2 A_2}{\partial\theta^2}\, E + \frac{1}{2}\,\mathrm{tr}\,\frac{\partial^2 A_3}{\partial\theta^2}\, E\, E + \mathrm{tr}\,\frac{\partial^2 A_4}{\partial\theta^2}\, E\,\xi + \frac{1}{2}\,\mathrm{tr}\,\frac{\partial^2 A_5}{\partial\theta^2}\,\xi\,\xi\Big]\overset{\circ}{\theta} + \frac{\partial A_1}{\partial\theta}\,\dot{\xi} + \frac{\partial A_2}{\partial\theta}\,\dot{E} + \mathrm{tr}\,\frac{\partial A_3}{\partial\theta}\,\dot{E}\, E + \mathrm{tr}\,\frac{\partial A_4}{\partial\theta}\,(E\,\overset{\circ}{\xi} + \dot{E}\,\xi) + \mathrm{tr}\,\frac{\partial A_5}{\partial\theta}\,\xi\,\overset{\circ}{\xi}\,.$$

Let us made the following final assumptions

$$(H5)\qquad \rho_o \frac{\partial A_3}{\partial\theta} = O(1)\,,\ \rho_o \frac{\partial A_4}{\partial\theta} = O(1)\,,\ \rho_o \frac{\partial A_2}{\partial\theta} = O(1),\ \rho_o \frac{\partial A_5}{\partial\theta} = O(1)\,,$$

$$\rho_o \frac{\partial A_1}{\partial\theta} = O(1)$$

$$\rho_o \frac{\partial^2 A_2}{\partial\theta^2} = O(1)\,,\ \rho_o \frac{\partial^2 A_3}{\partial\theta^2} = O(1)\,,\ \rho_o \frac{\partial^2 A_4}{\partial\theta^2} = O(1),\ \rho_o \frac{\partial^2 A_5}{\partial\theta^2} = O(1),$$

$$\rho_o \frac{\partial^2 A_1}{\partial\theta^2} = O(1)$$

then, recalling also (1.12),(1.23), we obtain the following linearization (in $\tilde{E},\tilde{\xi}$) of the energy equation (1.4 bis) :

$$(2.15)\qquad -\rho_o\,\theta\,\Big[\frac{\partial^2 A_o}{\partial\theta^2} + \frac{\partial^2 A_1}{\partial\theta^2}\,\tilde{\xi} + \frac{\partial^2 A_2}{\partial\theta^2}\,\tilde{E}\Big]\,\dot{\theta} - \rho_o\theta\,\Big[\frac{\partial A_1}{\partial\theta}\,\dot{\tilde{\xi}} + \frac{\partial A_2}{\partial\theta}\,\dot{\tilde{E}}\Big] =$$

$$= \mathrm{Div}(k\ \mathrm{Grad}\,\theta) + \rho_o r$$

2.5- Summing up the previous considerations we have done a linearization, only with respect to the infinitesimal strain, the hidden variables and the displacement u under the assumptions (H1),(H2),(H3),(H4),(H5).

In this way we have obtained the system of equations

$$(2.12)\qquad \mathrm{Div}\,[\tilde{S} + S_0] + \rho_0 f = \rho_0 \ddot{u}$$

$$(2.10)\qquad \tilde{S} + S_0 = B_0(X,\theta,g)\dot{\tilde{E}} + \rho_0 A_3(X,\theta)\tilde{E} + \rho_0 A_4(X,\theta)\,\tilde{\xi} + \rho_0 A_2(X,\theta)$$

$$(1.13)\qquad \dot{\tilde{\xi}} = -\,B_1(X,\theta,g)[\,A_1(X,\theta) + A_4(X,\theta)\tilde{E} + A_5(X,\theta)\tilde{\xi}\,]$$

$$(2.8)\qquad \rho_0[\,A_2(X,\theta_0) + A_4(X,\theta_0)\,\xi_0(X)] = S_0(X)$$

$$(2.15)\qquad -\,\rho_0\theta\,\Big[\frac{\partial^2 A_0}{\partial\theta^2} + \frac{\partial^2 A_1}{\partial\theta^2}\tilde{\xi} + \frac{\partial^2 A_2}{\partial\theta^2}\tilde{E}\Big]\dot{\theta} - \rho_0\theta\Big[\frac{\partial A_1}{\partial\theta}\dot{\tilde{\xi}} + \frac{\partial A_2}{\partial\theta}\dot{\tilde{E}}\Big] =$$

$$= \mathrm{Div}(k\,\mathrm{Grad}\theta) + \rho_0 r$$

$$(2.4)\qquad \tilde{E} = \frac{1}{2}\,[\mathrm{Grad}\,u + (\mathrm{Grad}\,u)^{*}] \qquad \dot{\tilde{E}} = \frac{1}{2}\,[\mathrm{Grad}\,\dot{u} + (\mathrm{Grad}\,\dot{u})^{*}]$$

to be completed with suitable initial and boundary conditions.

3- Some examples.

3.1- As a first example we shall take the case of thermoelasticity (example 1, § 1) with $S_o = 0$. From (2.10) we have

$$(3.1) \qquad \tilde{S} = \rho_o A_3(X,\theta)\,\tilde{E} + \rho_o B_2(X,\theta,\theta_o)\,(\theta - \theta_o)$$

with $\rho_o\, A_3(X,\theta) = K(X,\theta)$ (the stiffness) and

$$\rho_o\, B_2(X,\theta,\theta_o)(\theta - \theta_o) = K(X,\theta)\,\chi(X,\theta)(\theta - \theta_o),$$

χ is the thermal dilatation tensor. The only assumption to discuss is (H4).

If the variations of θ are small near the reference θ_o, i.e. $|\theta - \theta_o| = O(\delta)$, $|\dot{\theta}| = O(\delta)$, $|g| = O(\delta)$, $|\dot{g}| = O(\delta)$ then (H4) is satisfied. Moreover,

$$\frac{\partial A_2}{\partial \theta} = \frac{\partial B_2(\theta - \theta_o)}{\partial \theta} = K(\theta_o)\;\chi(\theta_o) + O(\delta)$$

A linearization of the energy equation (2.15) gives then the classical equations of the linear thermoelasticity. These equations are coupled by a term of the type $\theta_o\, K(\theta_o)\,\chi(\theta_o)\,\dot{\tilde{E}}$ in the energy equation ; fortunately for most applications the coupling can be neglected (see the example of BOLEY-WIENER| 5 |). Note that the heat equation is linear in θ .

In the case of <u>great temperature variations</u>, (H4) may also be verified ; it suffices that the product $K(X,\theta)\;\chi(X,\theta)\;(\theta - \theta_o)$ be small as it appears in some metals (see M. RAOUS |32|). Furthermore in this case the term $\frac{\partial^2 A_1}{\partial \theta^2}\tilde{E}$, which is of $O(\delta)$, is negligible with respect to the term $\frac{\partial^2 A_o}{\partial \theta^2}$ which is of $O(1)$ and, in the same way, the term $\rho_o \theta \frac{\partial A_2}{\partial \theta}\dot{\tilde{E}}$ is negligible with respect to the term $\rho_o \theta \frac{\partial^2 A_o}{\partial \theta^2}\dot{\theta}$.

The heat equation, which is nonlinear in θ is indeed uncoupled from the motion equation.

3.2- As second example we shall consider the Maxwell model of example 2, § 1. In this case from (1.14), (1.15), (1.22) and (1.24) we deduce that

$$\begin{aligned} B_1(X, \theta, g) &= \rho_o V^{-1}(X, \theta) \\ -\rho_o A_4(X,\theta) &= \rho_o A_5(X,\theta) = \rho_o A_3(X,\theta) = K(X,\theta) \\ \rho_o A_2(X,\theta) &= -\rho_o A_1(X,\theta) = K(X,\theta) e_o \quad \text{with } e_o = -\chi(X,\theta)(\theta - \theta_o) \end{aligned}$$

where we have $S_o = \xi_o = 0$ in the reference configuration. We see that the only hypotheses to be discussed are (H4) and (H2), which in this case are equivalent. Indeed the discussion can be done like in the example of thermoelasticity investigated in 3.1. In particular it appears that in the case of great temperature variations the nonlinear heat equation can be uncoupled from the motion equation.

This fact has also been pointed out by CROCHET and NAGHDI |13|.

4- Some results on the nonlinear heat equation.

4.1- Taking into account the examples of the previous section we shall at first study a nonlinear heat equation of the type

(4.1) $\rho_0 \, c\,(\theta)\,\dot{\theta} - \mathrm{Div}[\,k(X,\theta)\ \mathrm{Grad}\ \theta\,] = \rho_0 r_1(X,\theta) + \rho_0 r_2(X,t)$

subjected to the boundary conditions

(4.2)
$$\begin{cases} \theta(X,t) = g_0(X,t) & \text{on } \Gamma_0 \text{ (given temperature)} \\ n^{\mathsf{x}}.\ k(X,\theta)\ \mathrm{Grad}\ \theta = g_1(X,t) & \text{on } \Gamma_1 \text{ imposed flux)} \\ n^{\mathsf{x}}.\ k(X,\theta)\ \mathrm{Grad}\ \theta + a(\theta - g_2(X,t)) = 0 & \text{on } \Gamma_2 \text{ (radiation condition)} \end{cases}$$

and the initial condition

(4.3) $\theta(X,\ t_0) = \theta_0(X)$

Or, in the case where r_2, g_0, g_1, g_2 are T-periodic (T > 0) in time, the periodicity condition

(4.3 bis) $\theta(X,\ t) = \theta(X,\ t+T) \quad \forall(t,\ X)$;

$\Gamma_0, \Gamma_1, \Gamma_2$ are open subsets of the boundary $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma}_0 \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_2$.

4.2- The problems (4.1), (4.2), (4.3) and (4.1), (4.2), (4.3 bis) can be solved from the point of view of the nonlinear operator theory. Indeed we can apply theorems of § 5 of BARDOS-BREZIS | 2 |. If $g(X,t) = 0$(x), under very mild conditions of the type

(4.4) the elements $k_{i,j}(X,\eta)$ are bounded in η and measurable in $X \in \Omega$

(x) According to the trace theorems it is always possible to make the change of variable $\theta_1 = \theta - \theta_0$ where $\theta_0 = g_0$ on Γ_0 and then $\theta_1 = 0$ on Γ_0.

(4.5) $k(X,\eta)$ is positive definite in the sense

$$\sum_{i,j} k_{i,j}(X,\eta)\,\xi_i\,\xi_j \geqslant \alpha \sum_i \xi_i^2 \quad \text{with } \alpha > 0$$

(4.6) $r_1(X,\eta)$ is bounded in η and measurable in X or else

$r_1(X,\eta) = r_3(X)\eta + r_4(X)$ with $r_3(X) \leqslant 0$ and

$r_4 \in L^2(\Omega)$

(4.7) $a(X)$ in $(4.2)_3$ is measurable and non-negative,
it is not difficult to prove that the operator

$-\operatorname{Div}(k(X,\theta)\operatorname{Grad}\theta) - \rho_0 r_1(X,\theta)$, with (4.2) is an "operator of the calculus of variations" in the sense of LIONS [24 ; chap.2,§2] and so of type M (see e.g. Lions loc. cit.). For a proof of this type of results see AMIEL-GEYMONAT | 1 | and KENMOCHI |21| .

In the case of Cauchy problem additional deep results are obtained by LADYZENSKAJA-SOLONNIKOV-URALCEVA |22|.

5- Duality and virtual work.

5.1- Let Ω be a bounded connected open set in $\mathbb{R}^n$ (in practice one takes $n = 2$ or $n = 3$) with boundary $\partial\Omega$ sufficiently smooth. Let $\partial_1\Omega$ be a closed subset of $\partial\Omega$ with (n-1)-measure > 0. Let $\nu = (\nu_1,\ldots,\nu_n)$ be the unit normal to $\partial\Omega$ exterior to Ω . $H^1(\Omega)^n$ is the set of fields of isplacements $u = (u_1,\ldots,u_n)$ with $u_i \in H^1(\Omega)$, $i = 1,\ldots, n$; $H^1(\Omega)$ is the usual Sobolev space : for their properties see | 25 |.

If $u \in H^1(\Omega)^n$ then the trace $\gamma_0 u$ on $\partial\Omega$ is well-defined and $\gamma_0 u \in H^{1/2}(\partial\Omega)^n$; then $u_N = \sum_{i=1}^{n} \gamma_0 u_i \nu_i \in H^{1/2}(\partial\Omega)$ is the normal component of the trace of the displacement on $\partial\Omega$.

Let $\mathbb{V} = \left\{ v \in H^1(\Omega)^n \; ; \; \gamma_0 v = 0 \text{ on } \partial_1\Omega \text{ and } v_N = 0 \text{ on } \partial_2\Omega \right\}$

where $\partial_2\Omega$ is a closed subset of $\partial\Omega$ with (n-1)-measure $\geqslant 0$ (if meas $(\partial_2\Omega) = 0$ then the condition $v_N = 0$ must be dropped in the definition of $\mathbb{V}$), and let $\mathbb{V}$ be equiped with the hilbertian structure induced by $H^1(\Omega)^n$.

Let $\mathbb{E}$ be the space of infinitesimal tensor strain fields, i.e. of symmetric matrices $e = (e_{ij})_{i,j=1,..,n}$ with $e_{ij} \in L^2(\Omega)$ and let $\mathbb{S}$ be the space of tensor stresses fields, i.e. of symmetric matrices $s = (s_{ij})_{i,j=1,..,n}$ with $s_{i,j} \in L^2(\Omega)$. The spaces $\mathbb{E}$ and $\mathbb{S}$ form a dual system with the separating bilinear form

$$(5.1) \qquad \langle e , s \rangle = \sum_{i,j=1}^{n} \int_\Omega e_{ij}(x)\, s_{ij}(x)\, dx$$

which represents, from a mechanical point of view, the opposite of the work of the stress s in the deformation e . From a mathematical point of view $\mathbb{E}$ may be identified to $\mathbb{S}$, and then (5.1) represents the scalar product ; we shall denote by $\|\cdot\|$ the corresponding norm in $\mathbb{E}$ or $\mathbb{S}$.

The load space $\mathbb{L}$ and the space $\mathbb{V}$ are in duality with respect to the separating bilinear form $\langle\langle v , \phi \rangle\rangle$ which represents the work of the strength ϕ under the displacement v ; if $\phi = (f,h)$, where f is a regular volume force distributed in Ω and h a regular surface force on $\partial\Omega \setminus \partial_1\Omega$ having only a tangential component on $\partial_2\Omega$ (this means that $\sum_i h_i(x)\, \nu_i(x) = 0$ for a.e. $x \in \partial_2 \Omega$), then

(5.2) $$\ll v, \phi \gg \; = \; \sum_i \int_\Omega v_i(x) f_i(x)\,dx + \sum_i \int_{\partial\Omega\backslash\partial_1\Omega} \gamma_0 v_i(x)\, h_i(x)\, d\sigma$$

It is easy to see that this formula is true when $f_i \in L^2(\Omega)$ and $h_i \in L^2(\partial\Omega\backslash\partial_1\Omega)$ but its validity can be extended to a much more general situation, at least when both boundaries of $\partial_1\Omega$ and $\partial_2\Omega$ are regular in $\partial\Omega$.

D will denote the symmetric gradient operator

(5.3) $$Du = \left(\frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right)_{i,j = 1,..,n}$$

It is a linear continuous operator from $H^1(\Omega)^n$ into $\mathbb{E}$.
Thanks to Korn's inequality and to the fact that meas $(\partial_1\Omega) > 0$, D is a one-to-one bicontinuous mapping from $\mathbb{V}$ onto $D\mathbb{V}$ and $D\mathbb{V}$ is closed in $\mathbb{E}$ (see e.g. DUVAUT-LIONS |14 |chap. 3).

Let ${}^tD$ denote the transpose of D, defined by

(5.4) $$< Dv , s > \; = \; \ll v , {}^tDs \gg \qquad \forall v \in \mathbb{V} , \; \forall s \in \mathbb{S}$$

It is easy to see that ${}^tD$ is linear, continous and onto ; formally ${}^tDs = \phi$ means (we use the following classical notations : $s_N = \sum_{i,j} s_{ij}\nu_i\nu_j$, $s_{iT} = \sum_{j=1}^n s_{ij}\nu_j - s_N \nu_i$ and $s_T = (s_{iT})$)

(5.5) $$- \sum_{j=1}^n \frac{\partial}{\partial x_j} s_{ij} = f_i \qquad \text{in } \Omega$$

(5.6) $$\sum_{j=1}^n s_{ij}\,\nu_j = h_i \qquad \text{on } \partial\Omega \backslash (\partial_1\Omega \cup \partial_2\Omega)$$

(5.7) $$s_{iT} = h_i \qquad \text{on } \partial_2\Omega$$

and the methods of LIONS-MAGENES |25| render this interpretation rigorous.

For a more detailed analysis of the duality and the virtual work principle, see MOREAU |26|, NAYROLES |28|.

6- A viscoelastic constitutive equation with periodic coefficients.

Let T be a positive number. Let us assume that $\theta(X,t)$ is the unique T-periodic solution of the T-periodic boundary value problem associated with the nonlinear heat equation (4.1) and let us consider the constitutive equation of Maxwell type as in example 2 of §1.4, i.e.

$$(6.1)\qquad \begin{aligned} \tilde{S} &= K(X, \theta(X,t))\,[\tilde{E} + e_0 - \xi\,] \\ \tilde{S} &= V(X, \theta(X,t))\,\dot{\xi} \end{aligned}$$

where $e_0 = -\chi(X,\theta)(\theta - \theta_0)$ is the thermal dilatation field which corresponds to a non-stressed state in the reference configuration $\Omega(S_0 = 0)$. For simplicity we put

$$(6.2)\qquad K(X,t) = K(X, \theta(X,t)) \;,\quad V(X,t) = V(X, \theta(X,t))$$

$$(6.3)\qquad s = \tilde{S} \text{ (The total stress)}$$

$$(6.4)\qquad e = \tilde{E} + e_0 \text{ (The total strain)}$$

and then

$$(6.5)\qquad \begin{aligned} s &= K(X,t)\,[\,e - \xi\,] \\ s &= V(X,t)\,\dot{\xi} \end{aligned}$$

We assume :

A1- <u>K(X,t) and V(X,t) are symmetrical fourth order tensor, measurable and bounded on $\Omega \times R$ and such that for almost all (X,t)</u>

$$(6.6)\qquad \begin{aligned} K_{ij\ell m} &= K_{ji\ell m} = K_{ijm\ell} = K_{\ell mij} \\ V_{ij\ell m} &= V_{ji\ell m} = V_{ijm\ell} = V_{\ell mij} \end{aligned}$$

$$(6.7)\qquad K(X,t) = K(X, t+T) \qquad V(X,t) = V(X, t+T)$$

<u>and there exist $0 < \underline{k} \leqslant \bar{k}$, $0 < \underline{v} \leqslant \bar{v}$ such that for all symmetrical matrices (ν_{ij})</u>

$$(6.8)\quad \begin{aligned} &\underline{k} \sum_{i,j} \nu_{ij}^2 \leqslant \sum_{ij\ell m} K_{ij\ell m}(X,t)\, \nu_{ij}\, \nu_{\ell m} < k \sum_{ij} \nu_{ij}^2 \\ &\underline{v} \sum_{i,j} \nu_{ij}^2 \leqslant \sum_{ij\ell m} V_{ij\ell m}(X,t)\, \nu_{ij}\, \nu_{\ell m} \leqslant v \sum_{ij} \nu_{ij}^2 \end{aligned}$$

A2- The differential system

$$(6.9)\qquad \frac{dy}{dt} + V^{-1}(X,t)\, K(X,t)\, y = 0$$

is uniformely (in $X \in \Omega$) exponentially stable ∎

Let $A(X,t,\tau)$ be the fondamental resolvant of (6.9) such that $A(X,\tau,\tau) = j =$ Identity. From A2 we have for almost all X

$$(6.10)\qquad |\, A(X,t,\tau)\, | \leqslant c_1 \exp(-c_2(t-\tau)) \qquad \text{for } t \geqslant \tau$$

where c_1, c_2 are some positive constants (x). Furthermore from (6.5), (6.10) we can obtain and integral correspondance (xx) between s and e of the form

$$(6.11)\qquad s(X,t) = \int_{-\infty}^{t} F(X,t,\tau)\, [\, e(X,t) - e(X,\tau)\,]\, d\tau$$

where $F(X,t,\tau) = \frac{\partial G}{\partial \tau}(X,t,\tau)$, $G(X,t,\tau) = K(X,t)\, A(X,t,\tau)$ provided the initial $\xi_0(X)$ results from a "past" strain history $\hat{e}$, such that : $\xi_0(X) = \int_{-\infty}^{t_0} A(X,t_0,\tau)\, \hat{e}(\tau)\, d\tau$. We remark that

$$(6.12)\qquad F(X,t+T,\tau+T) = F(X,t,\tau)$$

Obviously, we can also (formally) write the inverse form :

$$(6.13)\qquad e(X,t) = \xi_0(X) + K^{-1}(X,t)\, s(X,t) + \int_{t_0}^{t} V^{-1}(X,\nu)\, s(X,\nu)\, d\nu$$

(x) In fact, if $V^{-1}K$ is symmetric, then A1 $\Longrightarrow$ A2 with $c_1 = 1$ and $c_2 = \underline{k} / \overline{v}$.

(xx) with aging.