Mathématiques et Applications 73

Serge Cohen
Jacques Istas

Fractional Fields and Applications

SMAI

Springer

Mathématiques et Applications

Directeurs de la collection :
J. Garnier et V. Perrier

73

For further volumes:
http://www.springer.com/series/2966

MATHÉMATIQUES & APPLICATIONS

Comité de Lecture 2012–2015/Editorial Board 2012–2015

Directeurs de la collection:
J. GARNIER et V. PERRIER

Serge Cohen · Jacques Istas

Fractional Fields and Applications

Springer

Serge Cohen
Institut de Mathématiques de Toulouse
Université Paul Sabatier
Toulouse
France

Jacques Istas
Laboratoire Jean Kuntzmann
Université de Grenoble et CNRS
Grenoble
France

ISSN 1154-483X
ISBN 978-3-642-36738-0 ISBN 978-3-642-36739-7 (eBook)
DOI 10.1007/978-3-642-36739-7
Springer Heidelberg New York Dordrecht London

Library of Congress Control Number: 2013933026

Mathematics Subject Classification (2010): 60G18, 60G22, 62M40, 65C99

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Foreword

In May 1990, I attended the annual day of the French Mathematical Society (SMF), which, that year, had chosen the crisp new subject of wavelet analysis. The flow of wavelets had not yet spread among large areas of Science, but a few top researchers were already actively working in this field. The SMF was aware of its great potential, both inside mathematics and for other sciences, and had decided to present it to its members.

At the end of the afternoon, a short man with a brushy moustache addressed me. He explained that he was a probabilist studying Gaussian fields, had heard that I was a student of Yves Meyer (the best possible recommendation…) and wished to enroll me in his research project. I had a very light background in probability, and was rather taken aback. But, if I had learned anything from my short mathematical life as a student in the starting subject of wavelets, it certainly was that big steps forward are the consequence of chance and unexpected encounters… and I jumped in without second thoughts. The next step was a couple of weeks at Clermont-Ferrand, where Albert Benassi explained his program. Albert was not a Bourbaki-style mathematician, but he was far-sighted, and had caught sight of a rich land, which would later become a fruitful field of interactions for mathematicians, signal analysts, and image processors. When Albert was doing mathematics, he was wearing seven-league boots, and did not clear all the way between each step; so following him in this adventure was a challenge… and a very rewarding one! When I joined, Daniel Roux, then Albert's Ph.D. student, was already part of the team. Serge Cohen and Jacques Istas would join a little later, and would quickly become prominent contributors in the development of this new area of mathematics. So that the present book can certainly be described as the final, polished output of the adventure that we lived as young scientists.

One should not infer that it only is an account of 20 years of exciting mathematics. It is much more: An introduction to a subject which now is extremely active, and flourishing in many directions. The conjunction of recent high resolution data acquisition techniques and Internet has for consequence that large collections of high resolution signals and images, coming from many different areas of science, are now available. The challenge for signal and image processing is to store, transmit, treat, and classify these data, which requires the introduction of wider and wider classes of sophisticated models. Their mathematical properties

have to be investigated, simulations are needed in order to confront visually models with data, and statistical tools need to be developed so that the corresponding key parameters of the models can be identified. This ambitious scientific program precisely is the purpose of this book.

A first originality is that it deals with fields, and not processes. If the literature concerning stochastic processes is extremely rich, it is much less the case for fields; this may be due to the fact that several key probabilistic tools have been developed specifically in dimension one, as a consequence of the natural interpretation of the one-dimensional variable as the time axis, and have less natural extensions in several dimensions. However, the need for a similar treatment of fields increased recently, with many motivations raising from image processing (in 2D), or for simulations of three-dimensional phenomena. A second originality is that the book encompasses the three facets of the same scientific field: A probabilistic study of classes of random fields, the development of statistical methods of identification of their parameters, and finally simulation techniques. These topics require diverse skills and usually are not met in the same book. However, each of them enriches the other two: Questions raised in one part motivate developments in another, and their conjunction will make this book an extremely valuable tool, both for mathematicians interested in understanding possible applications, and for scientists working in signal an image processing, and who want to master the mathematical background behind the models that they use; let us stress the fact that the very detailed and pedagogical chapter dealing with preliminaries make the book really accessible to scientists with a light background in probability and analysis.

This rich mixing of different aspects of the same subject certainly is in the spirit of the new way of performing scientific investigations which was initiated by Benoît Mandelbrot, half a century ago: His motivations to develop mathematical models rose from the inspection of data picked in a wide range of different sciences, and the mathematical properties of these models would often follow from observing their simulations; it is by no means a surprise that fractal analysis is a recurrent theme in this book. If the reader will allow me a bold comparison, the composition of the book with three views of the same scientific topics is reminiscent of some of the most famous Picasso portraits, where the juxtaposition of slightly different perspectives give a much deeper insight of the subject.

October 2011

Stéphane Jaffard,
Professor of Mathematics,
Université Paris Est Créteil

Contents

Notations

- a.s.: almost surely.
- $\overset{(a.s.)}{=}$: almost sure equality.
- $\#\Omega$: cardinal of the set Ω.
- $\mathcal{B}(K)$is the space of bounded functions on a set K.
- $\mathcal{C}^H$ is the space of Hölder-continuous functions on $[0,1]^d$.
- C^m is the space of m times differentiable functions.
- $C^{m,n}$ is the space of m times differentiable function in the first variable and n times differentiable function in the second variable.
- $det(A)$: determinant of the matrix A.
- $\overset{(d)}{=}$: equality in distribution.
- E_k: let $Z^{(d)}=\mathcal{N}_1(0,1)$ (notation defined below), then $E_k = \mathbb{E}|Z|^k$.
- i.i.d.: independent and identically distributed.
- $\mathbb{E}X$: expectation of the random variable X.
- $\widehat{f}(\xi) = \int_{\mathbb{R}}^d \exp(ix.\xi) f(x) \frac{dx}{(2\pi)^{d/2}}$
- $\langle f, g\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(\xi)\bar{g}(\xi) \frac{d\xi}{(2\pi)^{d/2}}$.
- We denote by $\Gamma(t) = \int_0^{+\infty} x^{t-1}e^{-x}dx$ the classical Gamma function for $t > 0$.
- iff : if and only if.
- K_X is the Reproducing Kernel Hilbert Space (RKHS) associated with a Gaussian field X.
- $\log_2(x) = \frac{\ln(x)}{\ln(2)}$.
- $\Lambda = \mathbb{Z} \times \mathbb{Z} \times \{1\}$
- $\Lambda^+ = \mathbb{N}^* \times \mathbb{Z} \times \{1\} \cup 0 \times \mathbb{Z} \times \{0\}$
- $\mathbb{N}$ the set of non-negative integers.
- $\mathbb{N}^*$ the set of positive integers.
- $\mathcal{N}_d(m, \Sigma)$: d-dimensional Gaussian vector of expectation m and covariance matrix Σ.
- $f(x) = O(g(x))$ as $x \to x_0$ when there exists a finite constant C such that $|f(x)| < C|g(x)|$ in a neighborhood of x_0. The point x_0 may be a real number, $+\infty$, or $-\infty$.

- $X_n = o_{\mathbb{P}}(Y_n)$ if $X_n,\ Y_n$ are sequences of random variables defined on the same probability space such that
$$\forall \epsilon > 0,\ \lim_{n\to\infty} \mathbb{P}(|X_n| \geq \epsilon |Y_n|) = 0.$$
This notation can be generalized to fields $X(x),\ Y(x)$as $X(x) = o_{\mathbb{P}}(Y(x))$ in a neighborhood of some x_0.
- $X_n = O_{\mathbb{P}}(Y_n)$ if $X_n,\ Y_n$ are sequences of random variables defined on the same probability space such that
$$\forall \epsilon > 0,\ \exists M > 0,\ \sup_{n\in\mathbb{N}} \mathbb{P}(|X_n| > M|Y_n|) < \epsilon.$$

This notation can be generalized to fields $X(x),\ Y(x)$as $X(x) = O_{\mathbb{P}}(Y(x))$ in a neighborhood of some x_0.

- $f(x) = o(g(x))$ as $x \to x_0$ when $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$;$x_0$ may be a real number or $+\infty$ or $-\infty$.
- $\mathbb{Q}$ is the set of rational numbers.
- $\mathbb{R}$ the set of real numbers.
- $\mathfrak{R}(z),\ \mathfrak{I}(z)$ are respectively the real part and the complex part of a complex number z.
- $R(x,y) = \mathbb{E}X_x\overline{X}_y$: covariance of a second-order centered field.
- r.v.: random variable.
- $\mathcal{S}$ is the space of fast decreasing functions.
- $\mathcal{S}'$ is the space of tempered distributions.
- $S^d = \{x \in \mathbb{R}^{d+1}$ s. t. $\|x\| = 1\}$, the d-dimensional sphere where $\|x\|$ is the Euclidean norm of $x \in \mathbb{R}^{d+1}$.
- s.t.: such that.
- $suppf$ is the support of the function f.
- $\sigma(X)$ is a sigma field on the set X.
- ${}^tA$: transpose of the matrix A.
- var$X$: variance of the random variable X.
- $u_n \sim v_n$ means that $\lim_{n\to+\infty} \dfrac{u_n}{v_n} = 1$.
- $u_n \asymp v_n$ means that there exists a constant $0 < C < \infty$ such that $\frac{1}{C} u_n \leq v_n \leq C u_n \quad \forall n \in \mathbb{N}$.
- $f(x) \sim g(x)$as $x \to x_0$ when $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$;$x_0$may be a real number or $+\infty$ or $-\infty$.
- $[x]$ is the integer such that $[x] \leq x < [x] + 1$.

Chapter 1
Introduction

Fractals everywhere! This is the title of a bestseller, but it is also a reality: Fractals are really everywhere. What a change since the days of Charles Hermite declaring "I turn away with fright and horror of this terrible scourge of continuous functions without derivative".[1] Historically, the first fractals are the Cantor set, and the Weierstrass function, followed by the famous Brownian motion. In these seminal examples, there were already between the lines the basic properties self-similarity and roughness, we will find throughout this book. But, what does mean this word "fractal"? Or its more or less synonymous "fractional"? There are probably as many definitions as there are people who work on the subject. We follow two tracks in this book.

- First, if an object is similar to each part of itself, i.e. the whole has the same shape as one or more of the parts.
- Second, if its Hausdorff dimension is not an integer.

A very common object is similar to each part of itself: a straight line! The self-similarity does not necessarily imply roughness. And, among the graphs of deterministic functions, the straight line is the only self-similar graph. A bit disappointing! But now replace condition "find a graph where each part is similar to itself" with the condition "find a graph where each part is *statistically* similar to itself". We obtain an infinite number of random graphs that satisfy this statistical self-similarity. In fact, there are so many that we do not know how to classify them, even if we know how to define some broad categories. For example, if we impose the graph to be Gaussian, for each self-similarity index $0 < H \leq 1$ there exists a unique solution, up to a constant, the famous fractional Brownian motions. For $H = 1$, the graph of fractional Brownian motion is a random Gaussian straight line. For $H \neq 1$, the graph of fractional Brownian motion is statistically self-similar, rough (its Hausdorff dimension is (a.s.) equal to 2-H and its pointwise Hölder exponent is H). Thus fractional Brownian motions became the archetypes of random fractals, we devote to fractional Brownian motions an important section in the Chap. 3, which is concerned

[1] "Je me détourne avec horreur et effroi de cette plaie lamentable des fonctions continues qui n'ont pas de dérivées" letter of Charles Hermite to Thomas Stieljtes, 20 may 1893.

S. Cohen and J. Istas, *Fractional Fields and Applications*,
Mathématiques et Applications 73, DOI: 10.1007/978-3-642-36739-7_1,

with self-similarity in a bigger generality. A smooth curve is locally approximated by a straight line, this is a general fact. A straight line is self-similar and a smooth curve becomes locally self-similar. Similarly, parts of a random graph could locally look like a, necessarily self-similar, random graph: One then speaks of local asymptotic self-similarity (in short class). For instance, numerous random fields look locally like fractional Brownian fields and we study several within the Chap. 4. To compare a model with reality implies the statistical estimation of the parameters describing the models. The estimation of the fractional parameter of a process is an issue for which there are several approaches known. In this book we will mainly use generalized quadratic variations to estimate fractional parameters. This statistical method is treated in the Chap. 5 devoted to Statistics. Finally, the simulation of theoretical models is a necessity. Again, it is not as simple as one might think, even in the Gaussian case. Numerical problems, computations time, are still not resolved, and we dedicate the Chap. 6 to discuss these issues.

Now that the broad picture is depicted we can be more precise and describe the points we stress in this book. Our main aim is to present multifractional fields. Surprisingly enough we don't write anywhere a definition of multifractional fields. Actually we think that multifractionality is a concept that depends on the applications one is interested in. Moreover we don't write any precise definition of a fractional field either. So it seems that the study of (multi)fractional fields is not rigorous in a mathematical sense. It is wrong and we want to provide in this introduction the reader with a walkthrough to multifractional fields.

If you have a good familiarity with Probability theory, Statistics, Analysis specialized in Hausdorff dimension and some background with wavelets then you can skip the Preliminaries Chap. 2. If not or if you discover later that you are missing some basic definition then this chapter is to make the book as self contained as possible for an undergraduate student in Mathematics. Please note that one does not need to be a mathematician to read this book. We have tried to make it as friendly as possible to applied scientists and engineers. Although most of the results of Chaps. 3 and 4 are given with complete and rigorous proofs, we have tried not be too technical. A nasty consequence of this choice is the fact that this book does not make a proper account to the most recent and sophisticated results in the domain of the (multi)fractional fields. On the other side if the first three chapters are still too technical for your own background, we may suggest to start with Chaps. 5 and 6.

The beginning of the hard core of the book is in Sect. 3.2.2, where the main properties of fractional Brownian process are explained. First of all you can find a uniqueness result that shows that up to a constant it is the only Gaussian process, which is H-self-similar and with stationary increments. In this part we make explicit the covariance structure, the regularity of the sample paths including the computation of the Hausdorff dimension of the graph, and two integral representations of fractional Brownian motion: One which is called the moving average representation and one which is in the Fourier domain and which is called harmonizable.

So far so good but we think that restricting the word "fractional" to either Gaussian, self-similar or processes with stationary increments is too restrictive for applications. Hence we will describe various generalizations of fractional Brownian motion in

Sect. 3.3 of Chap. 3. Some are parameterized by multidimensional index, some are non-Gaussian but they all keep in some sense the H-self similarity property, which is the common property of most of the objects considered in the Chap. 3. At this point we would like to prevent the reader from a possible misunderstanding: This book is a treaty neither on fractional Brownian motion nor on self-similarity. There are many good references in the literature on those topics (e.g. [53, 54]) and we will avoid to talk on any properties of fractional Brownian motion related to stochastic integration because as far as we know there are not easy to generalize to other (multi)fractional fields.

The goal of the Chap. 4 is to advertise multifractional fields. At this point a reasonable question is to wonder why bother with these generalizations of fractional Brownian motion. The answer comes from the data encountered by practitioners using fractional Brownian motion. Very often they have data for which the parameter H seems to vary with the time or the location where it is observed. It can be experienced because of the estimation of H or by simply observing the visual roughness of the sample paths. In this case the model of fractional Brownian motion is too rigid for your data because H is constant in this model or for a deeper mathematical reason: self-similarity and stationarity are global properties of the model and they cannot be adjusted to fit variations of data. A first solution to generalize the model of fractional Brownian motion is to localize the self-similarity property. This way of thinking yields a discussion carried in Sect. 4.2. In the Definition 4.2.1 of local asymptotic self-similarity the constant index H is replaced by a function $x \mapsto h(x)$ which varies with the location $x \in \mathbb{R}^d$ and which is called a multifractional function. However we do not think that local asymptotic self-similarity alone is enough to call a field multifractional. Obviously one wants to be able to compute at least some pointwise Hölder exponent at each point of multifractional fields. In the Chap. 4 we provide many examples of Gaussian locally asymptotically self-similar fields: Filtered white noises, multifractional Brownian fields, step fractional Brownian processes, generalized multifractional Gaussian processes. For each of this locally asymptotically self-similar field one can compute the multifractional function and its value at a location x is almost surely also the pointwise Hölder exponent. The reason why we have various models is that the assumptions on the regularity of the multifractional function $x \mapsto h(x)$ is less restrictive when a more sophisticated model is used. The previous list was ordered with growing sophistication. It is desirable because in some applications like image processing you expect discontinuous multifractional functions. Actually some data related to turbulence models have very wild "multifractional" functions for which the precise value of the pointwise Hölder exponent at a point x is not interesting but only the Hausdorff dimension of the set of the points x where the Hölder exponent is given is relevant. This theory yields "multifractal" processes or fields. It is not in the scope of this book since we are interested in applications where the pointwise Hölder exponent can be estimated. An introduction to multifractal processes is available in the Chap. 10 of [75]. Later in the Chap. 4 non-Gaussian multifractional fields are introduced using moving average or harmonizable integrals that extends the representations of fractional Brownian motion. Beside, new

models called fractional Lévy fields show that the multifractional function obtained in the lass property is not always equal to the pointwise Hölder exponent.

At this point we think that we can propose a rule of the thumb to know if a field should be called multifractional. *Multifractional fields have three properties:*

- *Local asymptotic self-similarity with a multifractional function h.*
- *The sample paths should have at every point x a local Hölder exponent, which can be different from h(x).*
- *Almost sure efficient estimation of the multifractional function h and of the local Hölder exponent should be feasible with the observation of only one sample of the fields.*

If you accept the previous rule you can understand why we are interested in the Chap. 5 in the estimation of the parameters of multifractional models. It is the third condition to have a useful fractional model. In practice the use of various discrete generalized quadratic variations along the paths of fractional fields described in the previous chapter is our favorite tool for the estimation. It allows in some sense a unified treatment for the estimation even if other techniques exist for fractional Brownian motion. In this chapter we have to distinguish the unifractional case where H is a real number and the multifractional case where we have to estimate the multifractional function h.

In the Chap. 6 we review some simulation techniques to get the flavor of what are the multifractional fields introduced in the previous chapters. Once again we do not focus on the simulation of fractional Brownian motion which is easily available in the literature (See [10] for instance.) but more on multifractional fields. Unfortunately many approximations have to be made to deal with computation times and this chapter should be thought more as an introduction to the problems encountered in simulating the multifractional fields than as the last words on this topic.

To finish this introduction we would like to thank many collaborators that help us to understand multifractional fields and to write this book. First of all both authors would like to express their endless gratitude to Albert Benassi. Albert has introduced both of us to the wonderful world of multifractional fields and the questions raised by their applications to real data. He was not only a collaborator for many of the articles used to write this book but also the source of most of the questions for which you have partial answers here.

Many parts of this book were taught at graduate students both in Toulouse and in Grenoble and we would like to thank the audience that helps us to improve our explanations. The universities where we are doing research and teaching leading to this book Université de Grenoble, Université de Toulouse, Université de Versailles Saint-Quentin en Yvelines and during a sabbatical semester of one of the author Cambridge University UK provided us stimulating environments. We are also indebted to many colleagues and former students. We gratefully acknowledge in particular Jean-Marc Azaïs, Jean-Marc Bardet, Alexandre Brouste, Jean-François Coeurjolly, Laure Coutin, Claire Christophe, Sébastien Déjean, Coralie Fritsch, Sébastien Gadat, Fabrice Gamboa, Stéphane Jaffard, Céline Lacaux, Sophie Lambert, Michel Ledoux, Olivier Perrin and Mario Wschebor.

Chapter 2
Preliminaries

In this chapter we have collected some results that will be used in the sequel of the book. We have divided these results in two parts. In the first one we recall some facts concerning stochastic processes. In the second part some results concerning fractal analysis are given.

2.1 Stochastic Fields

The main topic of this book is random fields. In this section we provide the reader with some theoretical background for random fields. The case of Gaussian fields, which give numerous examples, is particularly stressed. Nevertheless we assume some prerequisites for distributions of random variables and some elementary facts in probability theory. See [52] for a convenient introduction to probability theory.

2.1.1 Definition

Definition 2.1.1 *Stochastic field.*

Let T be a set and $(E, \mathcal{E})$ a measurable space. A stochastic field indexed by T, taking values in $(E, \mathcal{E})$, is a collection $(X_t, t \in T)$ of measurable maps X_t from a probability space $(\Omega, \sigma(\Omega), \mathbb{P})$ to $(E, \mathcal{E})$.

In this book, we are mainly concerned by real or complex valued stochastic fields indexed by $\mathbb{R}^d$, $d \geq 1$. If it is not mentioned otherwise, T will be $\mathbb{R}^d$ or a Borel subset in $\mathbb{R}^d$. When $d = 1$, one usually speaks of stochastic processes rather than of stochastic fields. The measurable space $(E, \mathcal{E})$ is the space $\mathbb{R}$ or $\mathbb{C}$ endowed with its Borel σ-algebra $\mathcal{B}(\mathbb{R})$ or $\mathcal{B}(\mathbb{C})$. For every $t \in T$, the stochastic field X yields a real or a complex valued random variable on $(\Omega, \sigma(\Omega), \mathbb{P})$. Fix now $\omega \in \Omega$. The map

S. Cohen and J. Istas, *Fractional Fields and Applications*,
Mathématiques et Applications 73, DOI: 10.1007/978-3-642-36739-7_2,

$t \to X_t(\omega)$ is the sample path of the stochastic field. We will mainly give conditions for real valued fields.

2.1.2 Kolmogorov's Consistency Theorem

The distribution of a stochastic field is characterized by finite dimensional distributions of the field.

Definition 2.1.2 *Finite dimensional distributions of the field.*

Two stochastic fields $(X_t, t \in T)$ and $(Y_t, t \in T)$ are said to be versions of each others if they have the same finite dimensional distributions. It means that for every $n \geq 1$ and $t_1, \ldots, t_n \in T$

$$(X_{t_1}, \ldots, X_{t_n}) \overset{(d)}{=} (Y_{t_1}, \ldots, Y_{t_n}). \tag{2.1}$$

These distributions are called the finite dimensional distributions (or margins) of the field $(X_t, t \in T)$.

Let us stress that the one dimensional distributions are not enough to characterize distributions of a stochastic field. Actually even for $n = 2$ we can have $X_{t_1} \overset{(d)}{=} Y_{t_1}$ and $X_{t_2} \overset{(d)}{=} Y_{t_2}$, whereas (X_{t_1}, X_{t_2}) has a different distribution from (Y_{t_1}, Y_{t_2}). Let $X_{t_1} \overset{(d)}{=} Y_{t_1} \overset{(d)}{=} X_{t_2} \overset{(d)}{=} Y_{t_2}$ be standard normal variables with $X_{t_1} = X_{t_2}$ and Y_{t_1} independent from Y_{t_2} then the distribution (X_{t_1}, X_{t_2}) is clearly different from the distribution of (Y_{t_1}, Y_{t_2}).

Let $(X_t, t \in T)$ be a stochastic field. The distribution of the random vector $(X_{t_1}, \ldots, X_{t_n})$, $n \geq 1$, $(t_1, \ldots, t_n) \in T^n$ is characterized by its characteristic function. For $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ the characteristic function is defined by

$$\psi_{t_1,\ldots,t_n}(\lambda_1, \ldots, \lambda_n) = \mathbb{E} \exp\left(i \sum_{j=1}^{n} \lambda_j X_{t_j} \right).$$

The Kolmogorov's consistency theorem [28] ensures the existence of a stochastic field such that the finite-dimensional distributions are consistent.

Theorem 2.1.1 *Kolmogorov's consistency theorem.*

Let $\psi_{t_1,\ldots,t_n}(\lambda_1, \ldots, \lambda_n)$, $n \geq 1$, $(t_1, \ldots, t_n) \in T^n$ be a collection of characteristic functions. Assume that for any permutation $(\sigma(1), \ldots, \sigma(n))$ of $(1, \ldots, n)$, one has

$$\psi_{t_{\sigma(1)},\ldots,t_{\sigma(n)}}(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}) = \psi_{t_1,\ldots,t_n}(\lambda_1, \ldots, \lambda_n), \tag{2.2}$$

and that, for any $m \leq n$

$$\psi_{t_1,\ldots,t_m}(\lambda_1,\ldots,\lambda_m) = \psi_{t_1,\ldots,t_n}(\lambda_1,\ldots,\lambda_m,0,\ldots,0). \tag{2.3}$$

The finite-dimensional distributions associated to these characteristic functions are called consistent. If the collection of characteristic functions is consistent, there exists a stochastic field such that, for $n \geq 1$, $(t_1,\ldots,t_n) \in T^n$,

$$\psi_{t_1,\ldots,t_n}(\lambda_1,\ldots,\lambda_n) = \mathbb{E}\exp\left(i\sum_{j=1}^{n}\lambda_j X_{t_j}\right). \tag{2.4}$$

Existence of the Brownian motion will be proved with the Kolmogorov's consistency theorem in the next section.

2.1.3 Gaussian Fields and Non-Negative Definite Functions

Many fields in this book will be Gaussian fields. First we recall some elementary facts concerning Gaussian random variables and Gaussian random vectors. The distribution of a Gaussian random variable with mean $\mathbb{E}X = m$ and variance σ^2 has a density

$$g_{m,\sigma^2}(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$$

with respect to Lebesgue measure. In this case the distribution of the Gaussian random variable is denoted by $\mathcal{N}(m,\sigma^2)$. Conventionally, if $X \overset{(a.s.)}{=} m \in \mathbb{R}$, the distribution of X is considered as a degenerated Gaussian distribution $\mathcal{N}(m,0)$. A Gaussian random variable is call standard if $X \overset{(d)}{=} \mathcal{N}(0,1)$. The characteristic function of $X \overset{(d)}{=} \mathcal{N}(m,\sigma^2)$ is

$$\mathbb{E}\exp(i\lambda X) = \exp\left(i\lambda m - \sigma^2\lambda^2/2\right).$$

Sometime we will need complex valued Gaussian variables.

Definition 2.1.3 *Let X be a complex valued random variable. It is a standard complex Gaussian variable if $\sqrt{2}\Re X$ and $\sqrt{2}\Im X$ are two independent real standard Gaussian random variables.*

Please note that if X is a standard complex Gaussian variable then

$$\mathbb{E}X = 0, \quad \mathbb{E}|X|^2 = 1,$$

this normalization explains the $\sqrt{2}$ in Definition 2.1.3.

Definition 2.1.4 *Let X be a complex valued random variable. It is a complex Gaussian variable if there is a standard complex Gaussian variable Z and $a, b \in \mathbb{C}$*

such that

$$X = aZ + b.$$

Definition 2.1.5 *A random vector $X = (X_1, \ldots, X_d)$ is called a Gaussian random vector if any finite linear combination of its coordinates $\sum_{i=1}^{d} \lambda_i X_i$ is a Gaussian random variable.*

One can check that if X is a real valued Gaussian random vector with mean $\mathbb{E}X = m \in \mathbb{R}^d$ and covariance matrix $\Gamma_{ij} = \mathbb{E}(X_i - m_i)(X_j - m_j)$, the characteristic function of X is given by

$$\mathbb{E}\exp(i\lambda.X) = \exp\left(i\lambda.m - \frac{1}{2}{}^t\lambda\Gamma\lambda\right),$$

where ${}^t\lambda$ is the transpose of the vector $\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_d \end{pmatrix}$. Let us recall that any covariance matrix is a non-negative symmetric matrix in the sense that $\forall\lambda \in \mathbb{R}^d,\ {}^t\lambda\Gamma\lambda \geq 0$. This classical non-negativity property for symmetric matrix is generalized to non-negative functions (Definition 2.1.9).

Example 2.1.1 *Existence of the Brownian motion.*

Let us now prove the existence of the standard Brownian motion with the Kolmogorov's consistency theorem. One aims to construct a stochastic process $(B(t))_{t\geq 0}$ satisfying the properties of the following definition.

Definition 2.1.6 *A process $(B(t))_{t\geq 0}$ satisfying*

- $B(0) = 0$,
- *for all $s \leq t$, the increment $B(t) - B(s)$ is independent of $(B(u), u \leq s)$,*
- *$B(t) - B(s)$ has a centered Gaussian distribution with variance $|t - s|$*

is called a standard Brownian motion.

Let $0 = t_0 \leq t_1 \leq \ldots \leq t_n$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Set $\Delta_j = B(t_j) - B(t_{j-1}),\ 1 \leq j \leq n$. Then

$$\sum_{j=1}^{n} \lambda_j B(t_j) = \sum_{j=1}^{n} \lambda_j \left(\sum_{k=1}^{j} \Delta_k\right).$$

Since the Δ_js have to be independent and Gaussian, the characteristic functions of the $B(t_j)$ should be given by

$$\mathbb{E}\exp\left(i\sum_{j=1}^{n}\lambda_j B(t_j)\right)=\exp\left(-1/2\sum_{j=1}^{n}\left(\sum_{k=j}^{n}\lambda_k\right)^2(t_j-t_{j-1})\right). \quad (2.5)$$

It is now straightforward that the characteristic function (2.5) satisfies the consistency conditions (2.2) and (2.3). By Kolmogorov's consistency theorem, the standard Brownian motion $(B(t), t \geq 0)$ exists and its distribution is unique. A process $(B(t), t \geq 0)$ such that $B(t) = aB_0(t) + b$, where $a, \ b \in \mathbb{R}$, and where B_0 is a standard Brownian motion is called a Brownian motion. The standardization in Definition 2.1.6 amounts to fix $B(0) = 0$ almost surely and $\mathrm{var}(B(1)) = 1$.

Definition 2.1.7 *Gaussian fields.*

A stochastic field $X_t, t \in T$ is a Gaussian field if and only if, for all $n \geq 1$, $t_1, \ldots, t_n \in T$, the random vector $(X_{t_1}, \ldots, X_{t_n})$ is a Gaussian random vector.

For Gaussian fields the distribution is characterized by the mean and the covariance functions. Let us define these functions.

Definition 2.1.8 *Mean value and covariance function.*

Let $(X_t, t \in T)$ be a complex valued stochastic field such that $\mathbb{E}|X_t|^2 < +\infty, \ \forall t \in T$. The mean value of X is the function $t \mapsto m(t) = \mathbb{E}X_t$. A real valued stochastic field such that $\mathbb{E}X_t = 0 \ \forall t \in T$ is called a centered field. The covariance function of X is the function

$$(t, s) \mapsto R(t, s) = \mathbb{E}(X_t - m(t))\overline{(X_s - m(s))}.$$

If X is a real valued field then the mean is also a real valued function and $R(t, s) = \mathbb{E}(X_t - m(t))(X_s - m(s))$.

We first give a characterization of covariance functions of Gaussian fields.

Definition 2.1.9 *Non-negative definite function.*

A complex valued function $(t, s) \to \psi(t, s), s, t \in T$ is called Hermitian if $\psi(s, t) = \overline{\psi(t, s)}$. Moreover an Hermitian function is a non-negative definite function if, for every $n \geq 1$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $t_1, \ldots, t_n \in T$

$$\sum_{i,j=1}^{n}\lambda_i\overline{\lambda_j}\psi(t_i, t_j) \geq 0.$$

If a function $(t, s) \to \psi(t, s), s, t \in T$ is real valued, then Hermitian functions are nothing else but symmetric functions.

Proposition 2.1.1 *Let ψ be a non-negative definite function. Then, for all $t, s \in T$,*

$$0 \leq \psi(t, t), \quad (2.6)$$

$$\Re\psi(t, s)^2 \leq \psi(t, t)\psi(s, s). \quad (2.7)$$

Proof of Proposition 2.1.1

Apply the definition of non-negative definite function with $n = 1$, $t_1 = t$ and $\lambda_1 = 1$, the Eq. (2.6) is straightforward.

Apply now the definition of non-negative definite function with $n = 2$, $t_1 = t$, $t_2 = s$, $\lambda_1 = 1$ and $\lambda_2 = -\lambda \in \mathbb{R}$. It follows that the polynomial $\lambda \to \lambda^2\psi(s,s) - 2\lambda\Re\psi(t,s) + \psi(t,t)$ is non-negative. The discriminant $4(\Re\psi(t,s)^2 - \psi(t,t)\,\psi(s,s))$ is therefore non-positive and (2.7) is proved.

Proposition 2.1.2 *The covariance function R of a stochastic field is a non-negative definite function.*

Proof of Proposition 2.1.2

Since

$$\sum_{j,j'=1}^{n} \lambda_j \overline{\lambda}_{j'} R(t_j, t_{j'}) = \mathbb{E}\left(|\sum_{j=1}^{n} \lambda_j (X(t_j) - \mathbb{E}X(t_j))|^2 \right) \geq 0,$$

the proposition is proved.

We now prove that any non-negative definite function is a covariance function of a Gaussian field. For the sake of simplicity we suppose that the fields are real valued until the end of the section.

Theorem 2.1.2 *Characterization of real valued Gaussian fields.*

Let $m(t), t \in T$ be a function and $R(t,s), t, s \in T$ be a non-negative definite function. Then the formula, for $n \geq 1$, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, $t_1, \ldots, t_n \in T$,

$$\mathbb{E}\exp\left(i \sum_{j=1}^{n} \lambda_j X_{t_j} \right) = \exp\left(i \sum_{j=1}^{n} \lambda_j m(t_j) \right) \times \exp\left(-1/2 \sum_{j,j'=1}^{n} \lambda_j \lambda_{j'} R(t_j, t_{j'}) \right)$$

characterizes the distribution of an unique real valued Gaussian field X, whose mean value function is m, and whose covariance function is R.

Proof of Theorem 2.1.2

In order to apply Kolmogorov's consistency theorem, we first need to check that the functions

$$\psi_{t_1,\ldots,t_n}(\lambda_1, \ldots, \lambda_n) = \exp\left(i \sum_{j=1}^{n} \lambda_j m(t_j) \right) \times \exp\left(-1/2 \sum_{j,j'=1}^{n} \lambda_j \lambda_{j'} R(t_j, t_{j'}) \right)$$

are characteristic functions. Let Σ_n be the matrix $(R(t_j, t_{j'}))_{j,j'=1,\dots,n}$. Σ_n is a symmetric, non-negative definite matrix. It can be written (e.g. [65]) in the form $\Sigma_n = P_n \Lambda_n P_n^{-1}$ where P_n is an orthogonal matrix and Λ_n is a diagonal matrix $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$ whose eigenvalues $\lambda_1, \dots, \lambda_n$ are non-negative. Define then $\sqrt{\Sigma_n}$ by $\sqrt{\Sigma_n} = P_n \sqrt{\Lambda_n} P_n^{-1}$ where $\sqrt{\Lambda_n} = \mathrm{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$. Let $\varepsilon_1, \dots, \varepsilon_n$ be n i.i.d. standard normal variables and set $E_n = {}^t(\varepsilon_1, \dots, \varepsilon_n)$. Set $M_n = {}^t(m(t_1), \dots, m(t_n))$. Then the characteristic function of the Gaussian random vector $Z_n = M_n + \sqrt{\Sigma_n} E_n$ is $\psi_{t_1,\dots,t_n}(\lambda_1, \dots, \lambda_n)$. It is then straightforward that these characteristic functions $\psi_{t_1,\dots,t_n}(\lambda_1, \dots, \lambda_n)$ satisfy the consistency conditions (2.3). By the Kolmogorov's consistency theorem, Theorem 2.1.2 is proved.

Non-negative functions are of prime importance for Gaussian fields. We therefore recall two characterizations (see [121] for Bochner's theorem and [127] for Schoenberg's theorem). First let us extend the Definition 2.1.9 of non-negative definite functions.

Definition 2.1.10 *A real valued function $f : T \mapsto \mathbb{R}$ is called non-negative definite if the function $(t, s) \to f(t - s)$ is non-negative definite in the sense of Definition 2.1.9 i.e. for every $n \geq 1$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, $t_1, \dots, t_n \in T$*

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j f(t_i - t_j) \geq 0 \text{ and } f(-t) = f(t)$$

Theorem 2.1.3 *Bochner's theorem.*

Among the continuous real valued functions, the non-negative definite functions on $T = \mathbb{R}^n$ are those functions which are the Fourier transforms of finite symmetric measures.

In Schoenberg's theorem functions of negative types are used.

Definition 2.1.11 *Functions of negative type.*

A real valued symmetric function $(t, s) \to \phi(t, s), s, t \in T$ is a function of negative type if, for every $n \geq 1$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} \lambda_i = 0$, $t_1, \dots, t_n \in T$

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \phi(t_i, t_j) \leq 0.$$

In Proposition 2.1.2 we remark that covariance functions are non-negative functions in the sense of Definition 2.1.9. If $R(t, s) = r(t - s)$ (which means that the centered stochastic fields X_t with covariance R is stationary in a weak sense c.f. Definition 2.1.22) then r is a non-negative function in the sense of Definition 2.1.10. In this case $\mathbb{E}(X(t) - X(s))^2 = \varphi(t, s)$ is of negative type.

Theorem 2.1.4 *Schoenberg's theorem.*

Let $(t, s) \to \phi(t, s), s, t \in T$ be a real valued symmetric continuous function with $\phi(t, t) = 0$.

1. *Fix $t_0 \in T$. Define ψ by*

$$\psi(t,s) = \phi(t_0,t) + \phi(t_0,s) - \phi(t,s).$$
$$\psi(s,t) = \Phi(t_0,s) + \phi(t_0,t) - \phi(s,t) = \psi(t,s).$$

   Then ϕ is a function of negative type, if and only if, ψ is a real valued non-negative definite function.
2. *ϕ is a function of negative type, if and only if, $e^{-\lambda\phi}$ is a non-negative definite function for all $\lambda \geq 0$.*

Let us consider some consequences of these theorems.

Corollary 2.1.1

- *Functions $(t,s) \to ||t||^{2H} + ||s||^{2H} - ||t-s||^{2H}, t, s \in \mathbb{R}^n$ are non-negative definite functions if and only if $0 < H \leq 1$.*
- *Functions $t \to e^{-|t|^\alpha}$, $t \in \mathbb{R}$ are characteristic functions if and only if $0 < \alpha \leq 2$.*

Proof of the Corollary 2.1.1

Function $(t,s) \to ||t-s||^2$ is of negative type. Indeed, for $\sum_{i=1}^{n} \lambda_i = 0$

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j ||t_i - t_j||^2 = -2||\sum_{i=1}^{n} \lambda_i t_i||^2 \leq 0.$$

Using the change of variables $u = \lambda x$, one easily proves that, for $x \geq 0, 0 < H < 1$

$$x^H = \tilde{C}_H \int_0^{+\infty} \frac{e^{-\lambda x} - 1}{\lambda^{1+H}} d\lambda, \tag{2.8}$$

where $\tilde{C}_H$ is a constant depending on H

$$\tilde{C}_H^{-1} = \int_0^{+\infty} \frac{e^{-u} - 1}{u^{1+H}} du.$$

Then

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j ||t_i - t_j||^{2H} = \tilde{C}_H \int_0^{+\infty} \frac{\sum_{i,j=1}^{n} \lambda_i \lambda_j e^{-\lambda||t_i - t_j||^2}}{\lambda^{1+H}} d\lambda.$$

By Schoenberg's theorem, $\sum_{i,j=1}^{n} \lambda_i \lambda_j e^{-\lambda||t_i - t_j||^2}$ is non-negative. Since $\tilde{C}_H < 0$, functions $(t, s) \to ||t - s||^{2H}$ are of negative type for $0 < H \leq 1$. Again by Schoenberg's theorem, functions $(t, s) \to ||t||^{2H} + ||s||^{2H} - ||t - s||^{2H}, t, s \in \mathbb{R}^n$ are non-negative definite functions for $0 < H \leq 1$. Let us now check that there are not non-negative definite for $H > 1$. Consider now three points t_1, t_2, t_3 on a straight line such that $\|t_2 - t_1\| = \|t_3 - t_2\| = 1$ and $\|t_3 - t_1\| = 2$. Take $\lambda_1 = \lambda_3 = -1$ and $\lambda_2 = 2$. Then $\sum_{i,j=1}^{3} \lambda_i \lambda_j ||t_i - t_j||^{2H} = -8 + 2^{2H+1}$ is strictly positive when $H > 1$. Functions $(t, s) \to ||t - s||^{2H}$ are not of negative type for $H > 1$. By Schoenberg's theorem, functions $(t, s) \to ||t||^{2H} + ||s||^{2H} - ||t - s||^{2H}, t, s \in \mathbb{R}^n$ are not non-negative definite functions for $H > 1$.

The part 2 of the Corollary follows then again by Schoenberg's theorem: functions $t \to e^{-|t|^\alpha}, t \in \mathbb{R}$ are non-negative definite functions if and only if $0 < \alpha \leq 2$. By Bochner's theorem, they are then characteristic functions. Actually, for $\alpha = 2$, it is the characteristic function of Gaussian random variable and for $0 < \alpha < 2$ it is the characteristic function of α-symmetric random variables (See for instance [126]).

2.1.4 Orthonormal Expansions of Gaussian Fields

The aim of this section is to present briefly the orthonormal expansion of Gaussian fields. One can read [1, 5, 109] for proofs and detailed results.

One can associate to every Gaussian field two Hilbert spaces. The first one is called the Gaussian space and it is a subspace of square integrable random variables. The second one is an Hilbert space of deterministic functions called Reproducing Kernel Hilbert Space. Let us first define Gaussian spaces.

Definition 2.1.12 *A vector space H of centered Gaussian random variables, which is a closed subspace of $L^2(\Omega, \mathcal{A}, P)$ is called a Gaussian space.*

Actually Gaussian spaces are generalizations of random vectors. Let us consider $(X_1, \ldots, X_d)$ a random Gaussian vector and

$$H = \{\sum_{i=1}^{d} \lambda_i X_i, \ \text{for} (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d\}.$$

It is clear that H is a finite dimensional Gaussian space.

In this section, we assume that Gaussian fields are complex valued. When the Gaussian field is real valued the following definition clearly makes sense, the reader has just to forget the conjugation bar on the second factor.

Definition 2.1.13 *Let $X_t, t \in T$ be a centered Gaussian field (i.e. $\mathbb{E}X_t = 0, \ \forall t \in T$). The subspace of $L^2(\Omega, \mathcal{A}, P)$ of the linear span of margins $X_t, \ \forall t \in T$ and of their*

limits in L^2 is denoted by

$$\mathcal{H}_X = \{Z \text{ such that } \exists n \in \mathbb{N},\ \exists \lambda_i \in \mathbb{C} \text{ for } i = 1, \ldots, n \text{ and } Z = \overline{\sum_{i=1}^{n} \lambda_i X_{t_i}}^{L^2}\} \quad , \tag{2.9}$$

where $\overline{E}^{L^2}$ is the closure in L^2 of the set E. The space

$$K_X = \{h_Z : T \mapsto \mathbb{C} \text{ such that } \exists Z \in \mathcal{H}_X \text{ and } h_Z(t) = \mathbb{E}\left(Z\overline{X_t}\right)\} \tag{2.10}$$

endowed with the Hermitian form

$$\langle h_{Z_1}, h_{Z_2} \rangle_{H_X} \stackrel{def}{=} \mathbb{E}\left(Z_1 \overline{Z_2}\right) \tag{2.11}$$

is called the Reproducing Kernel Hilbert Space (in short RKHS) of the Gaussian field X.

Let us remark that every random variable in the Gaussian space of a Gaussian field X is Gaussian since limits of Gaussian random variables are Gaussian.

Proposition 2.1.3 *If we use the notations of Definition 2.1.13 the map $h : \mathcal{H}_X \mapsto H_X$ defined by $Z \mapsto h_Z$ is a one to one linear map such that*

$$\|h_Z\|_{K_X} = \|Z\|_{\mathcal{H}_X} \quad \forall Z \in \mathcal{H}_X. \tag{2.12}$$

Proof of the Proposition 2.1.3

Obviously h is a linear map satisfying (2.12). Since $\mathcal{H}_X$ is the linear span of X_t, it is a one to one map. The isometry property is a consequence of the definition of the scalar product on the RKHS.

If X is a Gaussian field, please remark that

$$h_{X_t}(s) = \mathbb{E}(X_t \overline{X_s}) = R(t, s). \tag{2.13}$$

Hence finite linear combinations of the functions $(R(t, .))_{t \in T}$ are dense in H_X. The name of "reproducing" Kernel Hilbert space comes from the following property:

$$\forall h_Z \in K_X \quad \langle h_Z, R(t, .) \rangle_{K_X} = \mathbb{E}(Z\overline{X_t}) = h_Z(t). \tag{2.14}$$

In particular

$$\langle R(t, .), R(s, .) \rangle_{H_X} = R(t, s) \quad \forall s,\ t \in T. \tag{2.15}$$

Let us now suppose that there exists a countable orthonormal basis $(e_n)_n \in \mathbb{N}$ of K_X. One will consider two orthonormal series, one in the Gaussian space, the other one in the RKHS.

Theorem 2.1.5 *Let X be a centered Gaussian field and $(e_n)_{n\in\mathbb{N}}$ an orthonormal basis of K_X. Let us denote by $\eta_n = h^{-1}(e_n)$ a random variable in $\mathcal{H}_X$. Then the variables $(\eta_n)_{n\in\mathbb{N}}$ are independent identically standard Gaussian random variables and constitute an orthonormal basis of $\mathcal{H}_X$. Moreover*

$$\forall t \in T, \quad X_t = \sum_{n=0}^{+\infty} \eta_n \overline{e_n(t)} \tag{2.16}$$

where the convergence is in $L^2(\Omega)$ and

$$\forall t \in T \quad R(t,.) = \sum_{n=0}^{+\infty} e_n(.)\overline{e_n(t)} \tag{2.17}$$

where the convergence is in K_X.

Please remark that the convergence in (2.16) can be strengthened under mild assumptions to get an almost sure convergence in a functional sense, similarly in (2.17) the convergence is very often uniform. We refer to [5] for abstract results on those questions, but many examples will be found in the next chapter.

Proof of Theorem 2.1.5

Since $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of K_X, $(\eta_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}_X$. Since every linear combination of the η_n's are Gaussian random variables, orthogonality yields independence and $\|\eta_n\|_{L^2} = 1$ means that (η_n) is a standard Gaussian random variable. Moreover $X_t \in \mathcal{H}_X$ and

$$\begin{aligned} X_t &= \sum_{n=0}^{+\infty} \mathbb{E}(X_t \overline{\eta_n})\eta_n \\ &= \sum_{n=0}^{+\infty} \langle R(t,.), e_n\rangle_{K_X} \eta_n \\ &= \sum_{n=0}^{+\infty} \eta_n \overline{e_n}(t), \end{aligned}$$

where the convergence is in $\mathcal{H}_X$. With the same arguments

$$R(t,.) = \sum_{n=0}^{+\infty} \langle R(t,.), e_n\rangle_{H_X} e_n(.)$$

where the convergence is in K_X.

Let us give some elementary examples of Reproducing Kernel Hilbert Space.

First let us consider $T = \{1, \ldots, n\}$, then a real centered Gaussian field is nothing but a real valued centered Gaussian random vector $(X_1, \ldots, X_n)$. Let us consider

the covariance matrix $\mathcal{R} = (R(i,j))_{i,j\in T} = (\mathbb{E}(X_i\overline{X_j}))_{i,j\in T}$. Matrix $\mathcal{R}$ is a non-negative symmetric matrix. In this case both the Gaussian space and the RKHS are finite dimensional vector spaces. Every function f in K_X maps $\{1,\dots,n\}$ onto $\mathbb{R}$ and can be viewed as a vector $(f(1),\dots,f(n)) \in \mathbb{R}^n$. Since the linear span of $(R(i,.))_{i\in T}$ is K_X, there exist $\lambda_1,\dots,\lambda_n$ in $\mathbb{R}$ such that

$$f(.) = \sum_{i=1}^{n} \lambda_i R(i,.).$$

Moreover, if $g \in K_X$ is given by

$$g(.) = \sum_{j=1}^{n} \mu_j R(j,.),$$

then

$$\langle f, g\rangle_{K_X} = \sum_{i,j=1}^{n} \lambda_i \mu_j R(i,j).$$

In this case the dimension of K_X and $\mathcal{H}_X$ is the rank of the matrix $\mathcal{R}$.

Another example of Gaussian field is given by standard Brownian motion introduced in Example 2.1.1. For sake of simplicity let us take $T = [0,1]$. One can compute the covariance of this centered Gaussian field. If $s \le t$

$$\mathbb{E}(B_s B_t) = \mathbb{E}(B_s(B_s + B_t - B_s)) = s,$$

hence,

$$R(s,t) = \min(s,t).$$

Because of (2.15) we already know that

$$\langle R(t,.), R(s,.)\rangle_{K_X} = \min(s,t).$$

Remark that the derivative of $R(s,.)$ exists for every $t \neq s$, and $\forall s \in [0,1]$, $\frac{\partial R}{\partial u}(s,u) = \mathbf{1}_{[0,s]}(u)$. Then,

$$\langle R(t,.), R(s,.)\rangle_{K_X} = \int_0^1 \frac{\partial R}{\partial u}(s,u)\frac{\partial R}{\partial u}(t,u)du.$$

If $f(u) = \sum_{i=1}^{n} \lambda_i R(s_i,u)$ and $g(u) = \sum_{i=1}^{n'} \mu_i R(s_i,u)$ then