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Free Discontinuity Problems

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Introduction

The present volume collects the lectures notes of the courses given in July 2014 in the ERC "School on Free Discontinuity problems" that we organized at the Centro De Giorgi of the Scuola Normale Superiore at Pisa. The aim of the school was to present the main analytical and geometric ideas developed in the study of the so called free discontinuity problems by discussing three important examples: the Mumford-Shah model for image segmentation, a variational model for the epitaxial growth of thin films, and the sharp interface limit of an energy functional proposed by Ohta-Kawasaki to model pattern formation in dyblock copolymers. The common feature of these variational problems is the competition between volume and surface energies. The latter are concentrated on $(N-1)$-dimensional sets which are not given a priori and that indeed are the main unknown of the problem. They can be either rectifiable sets, as in the case of Mumford-Shah problem, or boundaries, as in the two other models.

The lectures were given by Matteo Focardi, Giovanni Leoni and Massimiliano Morini. They kindly agreed to write down in an extended form the content of their courses with the aim of both reviewing the main results of the theory and presenting the latest developments.

The volume starts with the contribution of Matteo Focardi "Fine regularity results for Mumford-Shah minimizers: porosity, higher integrability and the Mumford-Shah conjecture". He begins by presenting the weak formulation of the Mumford-Shah problem and the existence result of De Giorgi, Carriero and Leaci, which is showed by proving the equivalence between the weak and the strong formulation. In turn, this is an almost immediate consequence of a density lower bound for the jump set of the minimizer. Here, two different proofs of this density estimate are given, one due to De Lellis and Focardi and another one which follows from an almost monotonicity formula of Bucur and Luckhaus. Then, after recalling the partial regularity results proved in the '90s, the rest of the lecture focuses on the Mumford-Shah conjecture. The starting point is a result by Ambrosio, Fusco and Hutchinson which provides a quantitative link between the higher integrability of the gradient of a minimizer and the dimension of the singular set of the jump set. This result has been recently refined by De Lellis and Focardi who actually proved the equivalence between a weaker version of the Mumford-Shah conjecture and a precise degree of integrability of the gradient of the minimizers. A detailed proof of this equivalence is provided here. The last part of the lecture contains two higher integrability results. The first one, in two dimensions, is due to De Lellis and Focardi and it is based on Caccioppoli partitions. The second one, which applies to any dimension, was proved by De Philippis and Figalli using the porosity property of the jump set.

The lecture by Giovanni Leoni "Variational models for epitaxial growth" deals with a model for the epitaxial growth of thin films proposed by Spencer and Tersoff. They assume a planar symmetry of the three-dimensional configuration of the film. This leads to a two-dimensional model where the free surface of the film is represented as the graph of a periodic function over a reference interval and the substrate occupies an infinite strip. The total energy is given by the sum of two terms. The first one measures the length of the film weighed with a positive coefficient with a step discontinuity at the interface between the film and the substrate. The other one takes care of the elastic energy needed to deform the material. In the model the elastic properties of the film and the substrate are described by the same elasticity tensor. However, the interesting feature is the presence of a mismatch strain at the interface of the two materials. This mismatch is responsible of the so called islands formation, *i.e.*, non flat minimal configurations. The lecture starts with a detailed presentation of Korn's inequality in C^1 and Lipschitz domains which has its own interest besides the applications to thin films. Also of independent interest is the subsequent section which presents the classical results of Grisvard on the regularity of solutions of Lamé systems in polygonal domains. The second part of the lecture is devoted to the existence and regularity of minimizers. Here Leoni presents in an unified and comprehensive way various results scattered in the literature of the last fifteen years. This presentation can be very useful for young mathematicians interested in entering in a subject where mathematical research is still very active.

Massimiliano Morini presents a contribution on "Local and global minimality results for an isoperimetric problem with long-range interaction". The energy functional is given by the sum of the perimeter of a set E inside a fixed container Ω and the Dirichlet integral of the solution, under Neumann or periodic boundary conditions, of the Poisson equation with a right hand side depending on the characteristic function of E. When minimizing this energy under a volume constraint, one observes a wide class of configurations ranging from one or several almost spherical droplets to the union of cylinders or lamellae or even the more complicate patterns known as gyroids. The interesting feature of all these minimizers is that they are very close to periodic sets of constant mean curvature. Indeed various explicit constructions of stable critical configurations of the above type have been given in the last years. The lecture of Morini presents a second variation criterion for local minimality that he recently obtained in collaboration with Acerbi and Fusco. First, he derives the first and second variation formulae for the total energy established by Choksi and Sternberg. Then he proves that a strictly stable critical point, *i.e.*, a critical point with positive second variation, is a local minimizer under a volume constraint. To this aim he shows a quantitative estimate of the energy gap between the minimizer E and a competitor F in terms of the L^1 distance between the two sets. The second part of the lecture explains how this result can be applied to prove that certain lamellar and almost spherical configurations are indeed local and even global minimizers. Here one can find a unified approach to the recent results proved by Cicalese and Spadaro and by Morini and Sternberg.

We would like to conclude by thanking Matteo Focardi, Giovanni Leoni and Massimiliano Morini for preparing these very interesting lectures notes. We believe that they will be as successful as the beautiful courses that they gave two years ago at the school in Pisa.

Nicola Fusco
Aldo Pratelli

Fine regularity results for Mumford-Shah minimizers: porosity, higher integrability and the Mumford-Shah conjecture

Matteo Focardi

Abstract. We review some classical results and more recent insights about the regularity theory for local minimizers of the Mumford and Shah energy and their connections with the Mumford and Shah conjecture. We discuss in details the links among the latter, the porosity of the jump set and the higher integrability of the approximate gradient. In particular, higher integrability turns out to be related with an explicit estimate on the Hausdorff dimension of the singular set and an energetic characterization of the conjecture itself.

1 Introduction

The Mumford and Shah model is a prominent example of variational problem in image segmentation (see [69]). It is an algorithm able to detect the contours of the objects in a black and white digitized image. Representing the latter by a greyscale function $g \in L^\infty(\Omega, [0, 1])$, a smoothed version of the original image is then obtained by minimizing the functional

$$(v, K) \to \mathscr{F}(v, K, \Omega) + \gamma \int_{\Omega \setminus K} |v - g|^2 dx, \tag{1.1}$$

with

$$\mathscr{F}(v, K, \Omega) := \int_{\Omega \setminus K} |\nabla v|^2 \, dx + \beta \, \mathcal{H}^1(K), \tag{1.2}$$

where $\Omega \subseteq \mathbb{R}^2$ is an open set, K is a relatively closed subset of Ω with finite $\mathcal{H}^1$ measure, $v \in C^1(\Omega \setminus K)$, β and γ are nonnegative parameters to be tuned suitably according to the applications. In our discussion we can set $\beta = 1$ without loss of generality.

The role of the squared L^2 distance in (1.1) is that of a fidelity term in order that the output of the process is close in an average sense to the original input image g. The set K represents the set of contours of the objects in the image, the length of which is kept controlled by the penalization of its $\mathcal{H}^1$ measure to avoid over segmentation, while the Dirichlet energy of v favors sharp contours rather than zones where a thin layer of gray is used to pass smoothly from white to black or vice versa.

We stress the attention upon the fact that the set K is not assigned a priori and it is not a boundary in general. Therefore, this problem is not a free boundary problem, and new ideas and techniques had to be developed to solve it. Since its appearance in the late 80's to today the research on the Mumford and Shah problem, and on related fields, has been very active and different approaches have been developed. In this notes we shall focus mainly on that proposed by De Giorgi and Ambrosio. This is only due to a matter of taste of the Author and it is also dictated by understandable reasons of space. Even more, it is not possible to be exhaustive in our (short) presentation, therefore we refer to the books by Ambrosio, Fusco and Pallara [7] and David [26] for the proofs of many results we shall only quote, for a more detailed account of the several contributions in literature, for the many connections with other fields and for complete lists of references (see also the recent survey [53] that covers several parts of the regularity theory that are not presented here).

Going back to the Mumford and Shah minimization problem and trying to follow the path of the Direct Method of the Calculus of Variations, it is clear that a weak formulation calls for a function space allowing for discontinuities of co-dimension 1 in which an existence theory can be established. Therefore, by taking into account the structure of the energy, De Giorgi and Ambrosio were led to consider the space SBV of *Special functions of Bounded Variation*, *i.e.* the subspace of BV functions with singular part of the distributional derivative concentrated on a 1-dimensional set called in what follows the *jump set* (throughout the paper we will use standard notations and results concerning the spaces BV and SBV, following the book [7]).

The purpose of the present set of notes is basically to resume and collect several of the regularity properties known at present for Mumford and Shah minimizers. More precisely, Section 2 is devoted to recalling basic facts about the functional setting of the problem and its weak formulation. The celebrated De Giorgi, Carriero and Leaci [33] regularity result implying the equivalence between the strong and weak formulations, is discussed in details. In Subsection 2.3 we provide a recent proof by De Lellis and Focardi valid in the 2d case that gives an explicit constant in the density lower bound, and in Subsection 2.4 we discuss the almost monotonicity formula by Bucur and Luckhaus. Next, we state the Mumford and Shah conjecture. The understanding of such a claim is the goal at which researchers involved in this problem are striving for. In this perspective well-established and more recent fine regularity results on the jump set of minimizers are discussed in Section 3. Furthermore, we highlight two different paths that might lead to the solution in positive of the Mumford and Shah conjecture: the complete characterization of blow ups in Subsection 2.6 and a sharp higher integrability of the (approximate) gradient in Theorem 3.11 together with the uniqueness of blow up limits. In particular, we discuss in details the latter by following the ideas introduced by Ambrosio, Fusco and Hutchinson [4] linking higher integrability of the gradient of a minimizer with the size of the *singular set* of the minimizer itself, *i.e.* the subset of points of the jump set having no neighborhood in which the jump set itself is a regular curve. An explicit estimate shows that the bigger the integrability exponent of the gradient is, the lower the Hausdorff dimension of the singular set is (*cf.* Theorem 3.10). Pushing forward this approach, an energetic characterization of a slightly weaker form of the Mumford and Shah conjecture can be found beyond the scale of L^p spaces (*cf.* Theorem 3.11). In particular, the quoted estimate on the Hausdorff dimension of the full singular set reduces to the higher integrability property of the gradient and a corresponding estimate on a special subset of singular points: those for which the scaled Dirichlet energy is infinitesimal. The latter topic is dealt with in full details in Section 4 in the setting of Caccioppoli partitions as done by De Lellis and Focardi in [35]. The analysis of Section 4 allowed the same Authors to prove the higher integrability property in 2-dimensions as explained in Section 5. A different path leading to higher integrability in any dimension is to exploit the porosity of the jump set. This approach, due to De Philippis and Figalli [37], is the object of Section 7. Some preliminaries on porous sets are discussed in Section 6.

To conclude this introduction it is worth mentioning that the Mumford and Shah energy and the theory developed in order to study it, have been employed in many other fields. The applications to Fracture Mechanics, both in a static setting and for quasi-static irreversible crack-growth for brittle materials according to Griffith are important instances of that (see in particular [12], [7, Section 4.6.6] and [15,21,60]). It is also valuable to recall that several contributions in literature are devoted to the asymptotic analysis or the variational approximation of free discontinuity energies by means of De Giorgi's Γ-*convergence* theory. We refer to the books by Braides [13–15] for the analysis of several interesting problems arising from models in different fields (for a quick introduction to Γ-convergence see [40], for a more detailed account consult the treatise [20]).

The occasion to write this set of notes stems from the course "*Fine regularity results for Mumford-Shah minimizers: higher integrability of the gradient and estimates on the Hausdorff dimension of the singular set*" taught by the Author in July 2014 at Centro De Giorgi in Pisa within the activities of the "School on Free Discontinuity problems", ERC Research Period on Calculus of Variations and Analysis in Metric Spaces. The material collected here covers entirely the six lectures of the course, additional topics and some more recent insights are also included for the sake of completeness and clarity. It is a pleasure to acknowledge the hospitality of Centro De Giorgi and to gratefully thank N. Fusco and A. Pratelli, the organizers of the school, for their kind invitation. Let me also thank all the people in the audience for their attention, patience, comments and questions. In particular, the kind help of R. Cristoferi and E. Radici who read a preliminary version of these notes is acknowledged. Nevertheless, the Author is the solely responsible for all the inaccuracies contained in them.

2 Existence theory and first regularity results

In this section we shall overview the first basic issues of the problem. More generally we discuss the n-dimensional case, though we shall often make specific comments related to the 2-dimensional setting of the original problem (and sometimes to the 3d case as well). We shall freely use the notation for BV functions and Caccioppoli sets adopted in the book by Ambrosio, Fusco and Pallara [7]. We shall always refer to it also for the many results that we shall apply or even only quote without giving a precise citation.

2.1 Functional setting of the problem

A function $v \in L^1(\Omega)$ belongs to $BV(\Omega)$ if and only if Dv is a (vector-valued) Radon measure on the non empty open subset Ω of $\mathbb{R}^n$. The distributional derivative of v can be decomposed according to

$$Dv = \nabla v\, \mathcal{L}^n \llcorner \Omega + (v^+ - v^-)\nu_v\, \mathcal{H}^{n-1} \llcorner S_v + D^c v,$$

where

(i) ∇v is the density of the absolutely continuous part of Dv with respect to $\mathcal{L}^n \llcorner \Omega$ (and the *approximate gradient* of v in the sense of Geometric Measure Theory as well);
(ii) S_v is the set of *approximate discontinuities* of v, an $\mathcal{H}^{n-1}$-rectifiable set (so that $\mathcal{L}^n(S_v) = 0$) endowed with approximate normal ν_v for $\mathcal{H}^{n-1}$ a.e. on S_v;
(iii) $v^\pm$ are the *approximate one-sided traces* left by v $\mathcal{H}^{n-1}$ a.e. on S_v;
(iv) $D^c v$ is the rest in the Radon-Nikodym decomposition of the singular part of Dv after the absolutely continuous part with respect to

$\mathcal{H}^{n-1} \llcorner S_v$ has been identified. Thus, it is a singular measure both with respect to $\mathcal{L}^n \llcorner \Omega$ and to $\mathcal{H}^{n-1} \llcorner S_v$ (for more details see [7, Proposition 3.92]).

By taking into account the structure of the energy in (1.1), only volume and surface contributions are penalized, so that it is natural to introduce the following subspace of BV.

Definition 2.1 ([32], Section 4.1 [7]). $v \in BV(\Omega)$ is a *Special function of Bounded Variation*, in short $v \in SBV(\Omega)$, if $D^c v = 0$, *i.e.* $Dv = \nabla v\, \mathcal{L}^n \llcorner \Omega + (v^+ - v^-)\nu_v\, \mathcal{H}^{n-1} \llcorner S_v$.

No Cantor staircase type behavior is allowed for these functions. Simple examples are collected in the ensuing list:

(i) if $n = 1$ and $\Omega = (\alpha, \beta)$, $SBV\big((\alpha, \beta)\big)$ is easily described in view of the well known decomposition of BV functions of one variable. Indeed, any function in $SBV\big((\alpha, \beta)\big)$ is the sum of a $W^{1,1}\big((\alpha, \beta)\big)$ function with one of pure jump, *i.e.* $\sum_{i\in\mathbb{N}} a_i \chi_{(\alpha_i,\alpha_{i+1})}$, with $\alpha = \alpha_0$, $\alpha_i < \alpha_{i+1} < \beta$, $(a_i)_{i\in I} \in \ell^\infty$;

(ii) $W^{1,1}(\Omega) \subset SBV(\Omega)$. Clearly, $Dv = \nabla v\, \mathcal{L}^n \llcorner \Omega$. In this case ∇v coincides with the usual distributional gradient;

(iii) let $(E_i)_{i\in I}$, $I \subseteq \mathbb{N}$, be a *Caccioppoli partition* of Ω, *i.e.* $\mathcal{L}^n\big(\Omega \setminus \cup_i E_i\big) = 0$ and $\mathcal{L}^n(E_i \cap E_j) = 0$ if $i \neq j$, with the E_i's sets of finite perimeter such that
$$\sum_{i\in I} \mathrm{Per}(E_i) < \infty.$$
Then, $v = \sum_{i\in I} a_i\, \chi_{E_i} \in SBV(\Omega)$ if $(a_i)_{i\in I} \in \ell^\infty$. In this case, if $J_{\mathscr{E}} := \cup_i \partial^* E_i$ denotes the *set of interfaces* of $\mathscr{E}$, with $\partial^* E_i$ the *essential boundary* of E_i, then $\mathcal{H}^{n-1}(S_v \setminus J_{\mathscr{E}}) = 0$ and
$$Dv = (v^+ - v^-)\nu_v\, \mathcal{H}^{n-1} \llcorner J_{\mathscr{E}}.$$
Functions of this type have zero approximate gradient, they are called *piecewise constant* and form a subspace denoted by $SBV_0(\Omega)$ (*cf.* [7, Theorem 4.23]);

(iv) the function $v(\rho, \theta) := \sqrt{\rho} \cdot \sin(\theta/2)$ for $\theta \in (-\pi, \pi)$ and $\rho > 0$ is in $SBV(B_r)$ for all $r > 0$. In particular, $v \in SBV(B_r)\setminus\big(W^{1,1}(B_r) \oplus SBV_0(B_r)\big)$.

A general receipt to construct interesting examples of SBV functions can be obtained as follows (see [7, Proposition 4.4]).

Proposition 2.2. *If $K \subset \Omega$ is a closed set such that $\mathcal{H}^{n-1}(K) < +\infty$ and $v \in W^{1,1} \cap L^\infty(\Omega \setminus K)$, then $v \in SBV(\Omega)$ and*
$$\mathcal{H}^{n-1}(S_v \setminus K) = 0. \tag{2.1}$$

Clearly, property (2.1) above is not valid for a generic member of SBV, but it does for a significant class of functions: local minimizers of the energy under consideration (see below for the definition), actually satisfying even a stronger property (*cf.* Proposition 2.9).

2.2 Tonelli's Direct Method and Weak formulation

The difficulty in applying the Direct Method is related to the surface term for which it is hard to find a topology ensuring at the same time lower semicontinuity and pre-compactness for minimizing sequences. Using the Hausdorff local topology requires a very delicate study of the latter ones to rule out typical counterexamples as shown by Maddalena and Solimini in [56]. Here, we shall follow instead the original approach by De Giorgi and Ambrosio [32].

Keeping in mind the example in Proposition 2.2, the weak formulation of the problem under study is obtained naively by taking $K = S_v$. Loosely speaking in this approach the set of contours K is identified by the (Borel) set S_v of (approximate) discontinuities of the function v that is not fixed a priori. This is the reason for the terminology *free discontinuity* problem coined by De Giorgi. The (weak counterpart of the) Mumford and Shah energy $\mathscr{F}$ in (1.2) of a function v in $SBV(\Omega)$ on an open subset $A \subseteq \Omega$ then reads as

$$\mathscr{F}(v, A) = \mathrm{MS}(v, A) + \gamma \int_A |v - g|^2 dx, \tag{2.2}$$

where

$$\mathrm{MS}(v, A) := \int_A |\nabla v|^2 dx + \mathcal{H}^{n-1}(S_v \cap A). \tag{2.3}$$

For the sake of simplicity in case $A = \Omega$ we drop the dependence on the set of integration.

In passing, we note that, the class $\{v \in BV(\Omega) : Dv = D^c v\}$ of Cantor type functions is dense in BV w.r.to the L^1 topology, thus it is easy to infer that

$$\inf_{BV(\Omega)} \mathscr{F} = 0,$$

so that the restriction to SBV is needed in order not to trivialize the problem.

Ambrosio's SBV closure and compactness theorem (see [7, Theorems 4.7 and 4.8]) ensures the existence of a minimizer of $\mathscr{F}$ on SBV.

Theorem 2.3 (Ambrosio [2]). *Let* $(v_j)_j \subset SBV(\Omega)$ *be such that*

$$\sup_j \big(\mathrm{MS}(v_j) + \|v_j\|_{L^\infty(\Omega)}\big) < \infty,$$

then there exists a subsequence $(v_{j_k})_k$ *and a function* $v \in SBV(\Omega)$ *such that* $v_{j_k} \to v$ $L^p(\Omega)$, *for all* $p \in [1, \infty)$.

Moreover, we have the separated lower semicontinuity estimates

$$\int_\Omega |\nabla v|^2 dx \leq \liminf_k \int_\Omega |\nabla v_{j_k}|^2 dx \tag{2.4}$$

and

$$\mathcal{H}^{n-1}(S_v) \leq \liminf_k \mathcal{H}^{n-1}(S_{v_{j_k}}). \tag{2.5}$$

Ambrosio's theorem is the natural counterpart of Rellich-Kondrakov theorem in Sobolev spaces. Indeed, for Sobolev functions, it reduces essentially to that statement provided that an L^p rather than an L^∞ bound is assumed. More generally, Ambrosio's theorem holds true in the bigger space $GSBV$. In particular, (2.4) and (2.5) display a separate lower semicontinuity property for the two terms of the energy in a way that the two terms cannot combine to create neither a contribution for the other nor a Cantor type one.

By means of the chain rule formula for BV functions one can prove that the functional under consideration is decreasing under truncation, *i.e.* for all $k \in \mathbb{N}$

$$\mathscr{F}(\tau_k(v)) \leq \mathscr{F}(v) \quad \forall v \in SBV(\Omega),$$

if $\tau_k(v) := (v \wedge k) \vee (-k)$.

Therefore, being $g \in L^\infty(\Omega)$, we can always restrict ourselves to minimize it over the ball in $L^\infty(\Omega)$ of radius $\|g\|_{L^\infty(\Omega)}$. In conclusion, Theorem 2.3 always provides the existence of a (global) minimizer for the weak formulation of the problem.

Once the existence has been checked, necessary conditions satisfied by minimizers are deduced. Supposing $g \in C^1(\Omega)$, by means of internal variations, *i.e.* constructing competitors to test the minimality of u by composition with diffeomorphisms of Ω arbitrarily close to the identity of the type $\mathrm{Id} + \varepsilon\, \phi$, the Euler-Lagrange equation takes the form

$$\begin{aligned}\int_{\Omega\setminus S_u} \Big(\big(|\nabla u|^2 + \gamma (u-g)^2\big) \mathrm{div}\phi - 2\langle \nabla u, \nabla u \cdot \nabla \phi\rangle - 2\gamma(u-g)\langle \nabla g, \phi\rangle \Big) dx \\ + \int_{S_u} \mathrm{div}^{S_u}\phi \, d\mathcal{H}^{n-1} = 0\end{aligned} \tag{2.6}$$

for all $\phi \in C^1_c(\Omega, \mathbb{R}^n)$, $\mathrm{div}^{S_u}\phi$ denoting the tangential divergence of the field ϕ on S_u (*cf.* [7, Theorem 7.35]).

Instead, by using outer variations, *i.e.* range perturbations of the type $u + \varepsilon(v - u)$ for $v \in SBV(\Omega)$ such that $\mathrm{spt}(u - v) \Subset \Omega$ and $S_v \subseteq S_u$, we find

$$\int_\Omega \big(\langle \nabla u, \nabla(v - u)\rangle + \gamma\,(u - g)\,(v - u)\big)\,dx = 0. \tag{2.7}$$

2.3 Back to the strong formulation: the density lower bound

Existence of minimizers for the strong formulation of the problem is obtained via a regularity property enjoyed by (the jump set of) the minimizers of the weak counterpart. The results obtained in this framework will be instrumental also to establish way much finer regularity properties in the ensuing sections.

We start off analyzing the scaling of the energy in order to understand the local behavior of minimizers. This operation has to be done with some care since the volume and length terms in MS scale differently under affine change of variables of the domain. Let $v \in SBV(B_\rho(x))$, set

$$v_{x,\rho}(y) := \rho^{-1/2} v(x + \rho\, y), \tag{2.8}$$

then $v_{x,\rho} \in SBV(B_1)$, with

$$\mathrm{MS}(v_{x,\rho}, B_1) = \rho^{1-n}\mathrm{MS}(v, B_\rho(x))$$

and

$$\int_{B_1} |v_{x,\rho} - g_{x,\rho}|^2 dz = \rho^{-1-n} \int_{B_\rho(x)} |v - g|^2 dy.$$

Thus,

$$\rho^{1-n}\Big(\mathrm{MS}(v, B_\rho(x)) + \int_{B_\rho(x)} |v - g|^2 dz\Big)$$
$$= \mathrm{MS}(v_{x,\rho}, B_1) + \rho^2 \int_{B_1} |v_{x,\rho} - g_{x,\rho}|^2 dy.$$

By taking into account that $g \in L^\infty$ and that along the minimization process we are actually interested only in functions satisfying the bound $\|v\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)}$, we get

$$\rho^2 \int_{B_1} |v_{x,\rho} - g_{x,\rho}|^2 dy \leq 2\rho\, \|g\|^2_{L^\infty(\Omega)} = O(\rho) \qquad \rho \downarrow 0.$$

This calculation shows that, at the first order, the leading term in the energy $\mathscr{F}$ computed on $B_\rho(x)$ is that related to the MS functional, the other being a contribution of higher order that can be neglected in a preliminary analysis.

Motivated by this, we introduce a notion of minimality involving only the leading part of the energy. This corresponds to setting $\gamma = 0$ in the definition of $\mathscr{F}$ (*cf*. (2.2)).

Definition 2.4. A function $u \in SBV(\Omega)$ with $\mathrm{MS}(u) < \infty$[1] is a local minimizer of MS if

$$\mathrm{MS}(u) \leq \mathrm{MS}(v) \qquad \text{whenever } \{v \neq u\} \Subset \Omega.$$

In what follows, u will always denote a local minimizer of MS unless otherwise stated, and the class of all local minimizers shall be denoted by $\mathcal{M}(\Omega)$. Actually, we shall often refer to local minimizers simply as minimizers if no confusion can arise. In particular, regularity properties for minimizers of the whole energy can be obtained by perturbing the theory developed for local minimizers (see for instance Corollary 2.13 and Theorem 2.16 below).

Harmonic functions with small oscillation are minimizers as a simple consequence of (2.7).

Proposition 2.5 (Chambolle, see Proposition 6.8 [7]). *If u is harmonic in Ω', then $u \in \mathcal{M}(\Omega)$, for all $\Omega \Subset \Omega'$, provided*

$$\big(\sup_\Omega u - \inf_\Omega u\big)\|\nabla u\|_{L^\infty(\Omega)} \leq 1. \tag{2.9}$$

Proof. Let $A \Subset \Omega$. By Theorem 2.3 it is easy to show the existence of a minimizer $w \in SBV(\Omega)$ of the Dirichlet problem $\min\big\{\mathrm{MS}(v) : v \in SBV(\Omega),\ v = u \text{ on } \Omega \setminus A\big\}$. Moreover, by truncation $\inf_\Omega u \leq w \leq \sup_\Omega u$ $\mathcal{L}^n$ a.e. on Ω.

By the arbitrariness of A, the local minimality of u follows provided we show that $\mathrm{MS}(u, \Omega) \leq \mathrm{MS}(w, \Omega)$. To this aim, we use the Euler-Lagrange condition (2.7) with $\gamma = 0$, namely

$$\int_\Omega \langle \nabla w, \nabla(u - w)\rangle\, dx = 0 \Longleftrightarrow \int_\Omega |\nabla w|^2\, dx = \int_\Omega \langle \nabla w, \nabla u\rangle\, dx,$$

to get

$$\mathrm{MS}(u, \Omega) \leq \mathrm{MS}(w, \Omega) \Longleftrightarrow \int_\Omega \langle \nabla u, \nabla(u - w)\rangle\, dx \leq \mathcal{H}^{n-1}(S_w)$$

$$\Longleftrightarrow \int_\Omega \nabla u \cdot dD(u-w) - \int_{S_w} \langle \nabla u, \nu_w\rangle (w^+ - w^-) d\mathcal{H}^{n-1} \leq \mathcal{H}^{n-1}(S_w).$$

[1] The finite energy condition is actually not needed due to the local character of the notion introduced, it is assumed only for the sake of simplicity.

An integration by parts, the harmonicity of u and the equality $w = u$ on $\Omega \setminus A$ give

$$\int_\Omega \nabla u \cdot dD(u-w) = -\int_\Omega (u-w)\Delta u\, dx = 0,$$

and therefore

$$\mathrm{MS}(u,\Omega) \leq \mathrm{MS}(w,\Omega) \Longleftrightarrow -\int_{S_w} \langle \nabla u, \nu_w \rangle (w^+ - w^-) d\mathcal{H}^{n-1} \leq \mathcal{H}^{n-1}(S_w).$$

The conclusion follows from condition (2.9) as $\inf_\Omega u \leq w \leq \sup_\Omega u$ $\mathcal{L}^n$ a.e. on Ω. □

By means of the slicing theory in SBV, *i.e.* the characterization of SBV via restrictions to lines, one can also prove that *pure jumps*, *i.e.* functions as

$$a\chi_{\{\langle x-x_o,\nu\rangle>0\}} + b\chi_{\{\langle x-x_o,\nu\rangle<0\}} \tag{2.10}$$

for a and $b \in \mathbb{R}$ and $\nu \in \mathbb{S}^{n-1}$, are local minimizers as well (*cf.* [7, Proposition 6.8]]). Further examples shall be discussed in what follows (*cf.* Subsection 2.5).

As established in [33] in all dimensions (and proved alternatively in [23] and [25] in dimension two), if $u \in \mathcal{M}(\Omega)$ then the pair $(u, \Omega \cap \overline{S_u})$ is a minimizer of $\mathscr{F}$ for $\gamma = 0$. The main point is the identity $\mathcal{H}^{n-1}\big(\Omega \cap (\overline{S_u} \setminus S_u)\big) = 0$, which holds for every $u \in \mathcal{M}(\Omega)$. The groundbreaking paper [33] proves this identity via the following *density lower bound* estimate (see [7, Theorem 7.21]).

Theorem 2.6 (De Giorgi, Carriero and Leaci [33]). *There exist dimensional constants θ, $\varrho > 0$ such that for every $u \in \mathcal{M}(\Omega)$*

$$\mathrm{MS}(u, B_r(z)) \geq \theta\, r^{n-1} \tag{2.11}$$

for all $z \in \Omega \cap \overline{S_u}$, and all $r \in (0, \varrho \wedge \mathrm{dist}(z, \partial\Omega))$.

Building upon the same ideas, in [17] it is proved a slightly more precise result (see again [7, Theorem 7.21]).

Theorem 2.7 (Carriero and Leaci [17]). *There exist dimensional constants θ_0, $\varrho_0 > 0$ such that for every $u \in \mathcal{M}(\Omega)$*

$$\mathcal{H}^{n-1}(S_u \cap B_r(z)) \geq \theta_0\, r^{n-1} \tag{2.12}$$

for all $z \in \Omega \cap \overline{S_u}$, and all $r \in (0, \varrho_0 \wedge \mathrm{dist}(z, \partial\Omega))$.

In particular, from the latter we infer the so called *elimination property* for $\Omega \cap \overline{S_u}$, *i.e.* if $\mathcal{H}^{n-1}(S_u \cap B_r(z)) < \frac{\theta_0}{2^{n-1}} r^{n-1}$ then actually $\overline{S_u} \cap B_{r/2}(z) = \emptyset$.

Given Theorem 2.6 or 2.7 for granted we can easily prove the equivalence of the strong and weak formulation of the problem by means of the ensuing *density estimates*.

Lemma 2.8. *Let μ be a Radon measure on $\mathbb{R}^n$, B be a Borel set and $s \in [0, n]$ be such that*

$$\limsup_{r \downarrow 0} \frac{\mu(B_r(x))}{\omega_s r^s} \geq t \qquad \text{for all } x \in B.$$

Then, $\mu(B) \geq t\,\mathcal{H}^s(B)$.

Proposition 2.9. *Let $u \in \mathcal{M}(\Omega)$, then $\mathcal{H}^{n-1}\big(\Omega \cap (\overline{S_u} \setminus S_u)\big) = 0$. In particular $\big(u, \Omega \cap \overline{S_u}\big)$ is a local minimizer for $\mathscr{F}$ (with $\gamma = 0$).*

Proof of Proposition 2.9. In view of Theorem 2.7 we may apply the density estimates of Lemma 2.8 to $\mu = \mathcal{H}^{n-1} \llcorner S_u$ and to the Borel set $\Omega \cap (\overline{S_u} \setminus S_u)$ with $t = \theta_0$. Therefore, we deduce that

$$\theta_0 \mathcal{H}^{n-1}\big(\Omega \cap (\overline{S_u} \setminus S_u)\big) \leq \mu\big(\Omega \cap (\overline{S_u} \setminus S_u)\big) = 0.$$

Clearly, $\mathrm{MS}(u) = \mathscr{F}(u, \Omega \cap \overline{S_u})$, and the conclusion follows at once. □

The argument for (2.11) used by De Giorgi, Carriero and Leaci in [33], and similarly in [17] for (2.12), is indirect: it relies on Ambrosio's SBV compactness theorem and Poincaré-Wirtinger type inequality in SBV established in [33] (see also [7, Theorem 4.14] and [8, Proposition 2] for a version in which boundary values are preserved) to analyze blow up limits of minimizers (see Subsection 2.6 for the definition of blow ups) with vanishing jump energy and prove that they are harmonic functions (*cf.* [7, Theorem 7.21]). A contradiction argument shows that on small balls the energy of local minimizers inherits the decay properties as that of harmonic functions. Actually, the proof holds true for much more general energies (see [43], [7, Chapter 7]).

In the paper [34] an elementary proof valid only in 2-dimensions and tailored on the MS energy is given. No Poincaré-Wirtinger inequality, nor any compactness argument are required. Moreover, it has the merit to exhibit an explicit constant. Indeed, the proof in [34] is based on an observation of geometric nature and on a direct variational comparison argument. It also differs from those exploited in [23] and [25] to derive (2.12) in the 2-dimensional case.

Theorem 2.10 (De Lellis and Focardi [34]). *Let $u \in \mathcal{M}(\Omega)$. Then*

$$\mathrm{MS}(u, B_r(z)) \geq r \tag{2.13}$$

for all $z \in \Omega \cap \overline{S_u}$ and all $r \in (0, \mathrm{dist}(z, \partial\Omega))$. More precisely, the set $\Omega_u := \{z \in \Omega : (2.13)$ fails$\}$ is open and $\Omega_u = \Omega \setminus \overline{S_u}$[2]*.*

To the aim of establishing Theorem 2.10 we prove a consequence of (2.6), a monotonicity formula discovered independently by David and Léger in [27, Proposition 3.5] and by Maddalena and Solimini in [57]. The proof we present here is that given in [34, Lemma 2.1] (an analogous result holds true in any dimension with essentially the same proof).

Lemma 2.11. *Let $u \in \mathcal{M}(\Omega)$, $\Omega \subset \mathbb{R}^2$, then for every $z \in \Omega$ and for $\mathcal{L}^1$ a.e. $r \in (0, \mathrm{dist}(z, \partial\Omega))$*

$$\begin{aligned} r \int_{\partial B_r(z)} \left(\left(\frac{\partial u}{\partial \nu}\right)^2 - \left(\frac{\partial u}{\partial \tau}\right)^2 \right) d\mathcal{H}^1 + \mathcal{H}^1(S_u \cap B_z(r)) \\ = \int_{S_u \cap \partial B_r(z)} |\langle \nu_u^\perp(x), x\rangle| d\mathcal{H}^0(x), \end{aligned} \tag{2.14}$$

$\frac{\partial u}{\partial \nu}$ and $\frac{\partial u}{\partial \tau}$ being the projections of ∇u in the normal and tangential directions to $\partial B_r(z)$, respectively.[3]

Proof of Lemma 2.11. With fixed a point $z \in \Omega$, $r > 0$ with $B_r(z) \subseteq \Omega$, we consider special radial vector fields $\eta_{r,s} \in \mathrm{Lip} \cap C_c(B_r(z), \mathbb{R}^2)$, $s \in (0, r)$, in the first variation formula (2.6) (with $\gamma = 0$). Moreover, for the sake of simplicity we assume $z = 0$, and drop the subscript z in what follows. Let

$$\eta_{r,s}(x) := x\, \chi_{[0,s]}(|x|) + \frac{|x| - r}{s - r} x\, \chi_{(s,r]}(|x|),$$

then a routine calculation leads to

$$\nabla \eta_{r,s}(x) := \mathrm{Id}\, \chi_{[0,s]}(|x|) + \left(\frac{|x| - r}{s - r} \mathrm{Id} + \frac{1}{s - r} \frac{x}{|x|} \otimes x \right) \chi_{(s,r]}(|x|)$$

$\mathcal{L}^2$ a.e. in Ω. In turn, from the latter formula we infer for $\mathcal{L}^2$ a.e. in Ω

$$\mathrm{div}\eta_{r,s}(x) = 2\, \chi_{[0,s]}(|x|) + \left(2\frac{|x| - r}{s - r} + \frac{|x|}{s - r} \right) \chi_{(s,r]}(|x|),$$

[2] Actually, the very same proof shows also that $\Omega_u = \Omega \setminus \overline{J_u}$, where J_u is the subset of points of S_u for which one sided traces exist. Recall that $\mathcal{H}^{n-1}(S_v \setminus J_v) = 0$ for all $v \in BV(\Omega)$.

[3] For $\xi \in \mathbb{R}^2$, $\xi^\perp$ is the vector obtained by an anticlockwise rotation.

and, if $\nu_u(x)$ is a unit vector normal field in $x \in S_u$, for $\mathcal{H}^1$ a.e. $x \in S_u$

$$\mathrm{div}^{S_u}\eta_{r,s}(x) = \chi_{[0,s]}(|x|) + \left(\frac{|x|-r}{s-r} + \frac{1}{|x|(s-r)}|\langle x, \nu_u^{\perp}\rangle|^2\right)\chi_{(s,r]}(|x|).$$

Consider the set $I := \{\rho \in (0, \mathrm{dist}(0, \partial\Omega)) : \mathcal{H}^1(S_u \cap \partial B_\rho) = 0\}$, then $(0, \mathrm{dist}(0, \partial\Omega)) \setminus I$ is at most countable being $\mathcal{H}^1(S_u) < +\infty$. If ρ and $s \in I$, by inserting η_s in (2.6) we find

$$\begin{aligned}
&\frac{1}{s-r}\int_{B_r\setminus B_s}|x||\nabla u|^2dx - \frac{2}{s-r}\int_{B_r\setminus B_s}|x|\left\langle \nabla u, \left(\mathrm{Id} - \frac{x}{|x|}\otimes\frac{x}{|x|}\right)\nabla u\right\rangle dx\\
&\quad= \mathcal{H}^1(S_u \cap B_s) + \int_{S_u\cap(B_r\setminus B_s)}\frac{|x|-r}{s-r}d\mathcal{H}^1\\
&\qquad+ \frac{1}{s-r}\int_{S_u\cap(B_r\setminus B_s)}|x|\left|\left\langle\frac{x}{|x|}, \nu_u^{\perp}\right\rangle\right|^2 d\mathcal{H}^1.
\end{aligned}$$

Next we employ Co-Area formula and rewrite equality above as

$$\begin{aligned}
&\frac{1}{s-r}\int_s^r \rho\, d\rho\int_{\partial B_\rho}|\nabla u|^2 d\mathcal{H}^1 - \frac{2}{s-r}\int_s^r \rho\, d\rho\int_{\partial B_\rho}\left|\frac{\partial u}{\partial\tau}\right|^2 d\mathcal{H}^1\\
&\quad= \mathcal{H}^1(S_u \cap B_s) + \int_{S_u\cap(B_r\setminus B_s)}\frac{|x|-r}{s-r}d\mathcal{H}^1\\
&\qquad+ \frac{1}{s-r}\int_s^r d\rho\int_{S_u\cap\partial B_\rho}|\langle x, \nu_u^{\perp}\rangle| d\mathcal{H}^0
\end{aligned}$$

where $\nu := x/|x|$ denotes the radial unit vector and $\tau := \nu^{\perp}$ the tangential one. Lebesgue differentiation theorem then provides a subset I' of full measure in I such that if $r \in I'$ and we let $s \uparrow t^-$ it follows

$$\begin{aligned}
&-r\int_{\partial B_r}|\nabla u|^2 d\mathcal{H}^1 + 2r\int_{\partial B_r}\left|\frac{\partial u}{\partial\tau}\right|^2 d\mathcal{H}^1\\
&\quad= \mathcal{H}^1(S_u \cap B_r) - \int_{S_u\cap\partial B_r}|\langle x, \nu_u^{\perp}\rangle| d\mathcal{H}^0.
\end{aligned}$$

Formula (2.14) then follows straightforwardly. □

We are now ready to prove Theorem 2.10.

Proof of Theorem 2.10. Given $u \in \mathcal{M}(\Omega)$, $z \in \Omega$ and $r \in (0, \mathrm{dist}(z, \partial\Omega))$ let

$$e_z(r) := \int_{B_r(z)}|\nabla u|^2 dx, \quad \ell_z(r) := \mathcal{H}^1(S_u \cap B_r(z)),$$

and

$$m_z(r) := \mathrm{MS}(u, B_r(z)), \quad h_z(r) := e_z(r) + \frac{1}{2}\ell_z(r).$$

Clearly, $m_z(r) = e_z(r) + \ell_z(r) \leq 2h_z(r)$, with equality if and only if $e_z(r) = 0$.

Introduce the set $S_u^\star$ of points $x \in S_u$ for which

$$\lim_{r\downarrow 0} \frac{\mathcal{H}^1(S_u \cap B_r(x))}{2r} = 1\,. \tag{2.15}$$

Since S_u is rectifiable, $\mathcal{H}^1(S_u \setminus S_u^\star) = 0$. Next let $z \in \Omega$ be such that

$$m_z(R) < R \qquad \text{for some } R \in (0, \mathrm{dist}(z, \partial\Omega)). \tag{2.16}$$

We claim that $z \notin S_u^\star$.

W.l.o.g. we take $z = 0$ and drop the subscript z in e, ℓ, m and h.

In addition we can assume $e(R) > 0$. Otherwise, by the Co-Area formula and the trace theory of BV functions, we would find a radius $r < R$ such that $u|_{\partial B_r}$ is a constant (*cf.* the argument below). In turn, u would necessarily be constant in B_r because the energy decreases under truncations, thus implying $z \notin S_u^\star$. We can also assume $\ell(R) > 0$, since otherwise u would be harmonic in B_R and thus we would conclude $z \notin S_u^\star$.

We start next to compare the energy of u with that of an harmonic competitor on a suitable disk. The inequality $\ell(R) \leq m(R) < R$ is crucial to select good radii.

Step 1*:* For any fixed $r \in (0, R - \ell(R))$, there exists a set I_r of positive length in (r, R) such that

$$\frac{h(\rho)}{\rho} \leq \frac{1}{2} \cdot \frac{e(R) - e(r)}{R - r - (\ell(R) - \ell(r))} \qquad \text{for all } \rho \in I_r\,. \tag{2.17}$$

Define $J_r := \{t \in (r, R) : \mathcal{H}^0(S_u \cap \partial B_t) = 0\}$. We claim the existence of $J_r' \subseteq J_r$ with $\mathcal{L}^1(J_r') > 0$ and such that

$$\int_{\partial B_\rho} |\nabla u|^2 d\mathcal{H}^1 \leq \frac{e(R) - e(r)}{R - r - (\ell(R) - \ell(r))} \qquad \text{for all } \rho \in J_r'. \tag{2.18}$$

Indeed, we use the Co-Area formula for rectifiable sets (see [7, Theorem 2.93]) to find

$$\begin{aligned}\mathcal{L}^1((r, R) \setminus J_r) &\leq \int_{(r,R)\setminus J_r} \mathcal{H}^0(S_u \cap \partial B_t)dt \\ &= \int_{S_u\cap(B_R\setminus\overline{B_r})} \left|\left\langle \nu_u^\perp(x), \frac{x}{|x|}\right\rangle\right| d\mathcal{H}^1(x) \leq \ell(R) - \ell(r).\end{aligned}$$