Vibrations and Waves

George C. King

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Contents

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VIBRATIONS AND WAVES

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Franz Mandl

(1923–2009)

This book is dedicated to Franz Mandl. I first encountered him as an inspirational teacher when I was an undergraduate. Later, we became colleagues and firm friends at Manchester. Franz was the editor throughout the writing of the book and made many valuable suggestions and comments based upon his wide-ranging knowledge and profound understanding of physics. Discussions with him about the various topics presented in the book were always illuminating and this interaction was one of the joys of writing the book.

Editors' Preface to the Manchester Physics Series

The Manchester Physics Series is a series of textbooks at first degree level. It grew out of our experience at the University of Manchester, widely shared elsewhere, that many textbooks contain much more material than can be accommodated in a typical undergraduate course; and that this material is only rarely so arranged as to allow the definition of a short self-contained course. In planning these books we have had two objectives. One was to produce short books so that lecturers would find them attractive for undergraduate courses, and so that students would not be frightened off by their encyclopaedic size or price. To achieve this, we have been very selective in the choice of topics, with the emphasis on the basic physics together with some instructive, stimulating and useful applications. Our second objective was to produce books which allow courses of different lengths and difficulty to be selected with emphasis on different applications. To achieve such flexibility we have encouraged authors to use flow diagrams showing the logical connections between different chapters and to put some topics in starred sections. These cover more advanced and alternative material which is not required for the understanding of latter parts of each volume.

Although these books were conceived as a series, each of them is self-contained and can be used independently of the others. Several of them are suitable for wider use in other sciences. Each Author's Preface gives details about the level, prerequisites, etc., of that volume.

The Manchester Physics Series has been very successful since its inception 40 years ago, with total sales of more than a quarter of a million copies. We are extremely grateful to the many students and colleagues, at Manchester and elsewhere, for helpful criticisms and stimulating comments. Our particular thanks go to the authors for all the work they have done, for the many new ideas they have contributed, and for discussing patiently, and often accepting, the suggestions of the editors.

Finally we would like to thank our publishers, John Wiley & Sons, Ltd, for their enthusiastic and continued commitment to the Manchester Physics Series.

F. K. Loebinger
F. Mandl
D. J. Sandiford
August 2008

Author’s Preface

Vibrations and waves lie at the heart of many branches of the physical sciences and engineering. Consequently, their study is an essential part of the education of students in these disciplines. This book is based upon an introductory 24-lecture course on vibrations and waves given by the author at the University of Manchester. The course was attended by first-year undergraduate students taking physics or a joint honours degree course with physics. This book covers the topics given in the course although, in general, it amplifies to some extent the material delivered in the lectures.

The organisation of the book serves to provide a logical progression from the simple harmonic oscillator to waves in continuous media. The first three chapters deal with simple harmonic oscillations in various circumstances while the last four chapters deal with waves in their various forms. The connecting chapter (Chapter 4) deals with coupled oscillators which provide the bridge between waves and the simple harmonic oscillator. Chapter 1 describes simple harmonic motion in some detail. Here the universal importance of the simple harmonic oscillator is emphasised and it is shown how the elegant mathematical description of simple harmonic motion can be applied to a wide range of physical systems. Chapter 2 extends the study of simple harmonic motion to the case where damping forces are present as they invariably are in real physical situations. It also introduces the quality factor Q of an oscillating system. Chapter 3 describes forced oscillations, including the phenomenon of resonance where small forces can produce large oscillations and possibly catastrophic effects when a system is driven at its resonance frequency. Chapter 4 describes coupled oscillations and their representation in terms of the normal modes of the system. As noted above, coupled oscillators pave the way to the understanding of waves in continuous media. Chapter 5 deals with the physical characteristics of travelling waves and their mathematical description and introduces the fundamental wave equation. Chapter 6 deals with standing waves that are seen to be the normal modes of a vibrating system. A consideration of the general motion of a vibrating string as a superposition of normal modes leads to an introduction of the powerful technique of Fourier analysis. Chapter 7 deals with some of the most dramatic phenomena produced by waves, namely interference and diffraction. Finally, Chapter 8 describes the superposition of a group of waves to form a modulated wave or wave packet and the behaviour of this group of waves in a dispersive medium. Throughout the book, the fundamental principles of waves and vibrations are emphasised so that these principles can be applied to a wide range of oscillating systems and to a variety of waves including electromagnetic waves and sound waves. There are some topics that are not required for other parts of the book and these are indicated in the text.

Waves and vibrations are beautifully and concisely described in terms of the mathematical equations that are used throughout the book. However, emphasis is always placed on the physical meaning of these equations and undue mathematical complication and detail are avoided. An elementary knowledge of differentiation and integration is assumed. Simple differential equations are used and indeed waves and vibrations provide a particularly valuable way to explore the solutions of these differential equations and their relevance to real physical situations. Vibrations and waves are well described in complex representation. The relevant properties of complex numbers and their use in representing physical quantities are introduced in Chapter 3 where the power of the complex representation is also demonstrated.

Each chapter is accompanied by a set of problems that form an important part of the book. These have been designed to deepen the understanding of the reader and develop their skill and self-confidence in the application of the equations. Some solutions and hints to these problems are given at the end of the book. It is, of course, far more beneficial for the reader to try to solve the problems *before* consulting the solutions.

I am particularly indebted to Dr Franz Mandl who was my editor throughout the writing of the book. He read the manuscript with great care and physical insight and made numerous and valuable comments and suggestions. My discussions with him were always illuminating and rewarding and indeed interacting with him was one of the joys of writing the book. I am very grateful to Dr Michele Siggel-King, my wife, who produced all the figures in the book. She constructed many of the figures depicting oscillatory and wave motion using computer simulation programs and she turned my sketches into suitable figures for publication. I am also grateful to Michele for proofreading

the manuscript. I am grateful to Professor Fred Loebinger who made valuable comments about the figures and to Dr Antonio Juarez Reyes for working through some of the problems.

George C. King

1

Simple Harmonic Motion

In the physical world there are many examples of things that vibrate or oscillate, i.e. perform periodic motion. Everyday examples are a swinging pendulum, a plucked guitar string and a car bouncing up and down on its springs. The most basic form of periodic motion is called simple harmonic motion (SHM). In this chapter we develop quantitative descriptions of SHM. We obtain equations for the ways in which the displacement, velocity and acceleration of a simple harmonic oscillator vary with time and the ways in which the kinetic and potential energies of the oscillator vary. To do this we discuss two particularly important examples of SHM: a mass oscillating at the end of a spring and a swinging pendulum. We then extend our discussion to electrical circuits and show that the equations that describe the movement of charge in an oscillating electrical circuit are identical in form to those that describe, for example, the motion of a mass on the end of a spring. Thus if we understand one type of harmonic oscillator then we can readily understand and analyse many other types. The universal importance of SHM is that to a good approximation many real oscillating systems behave like simple harmonic oscillators when they undergo oscillations of small amplitude. Consequently, the elegant mathematical description of the simple harmonic oscillator that we will develop can be applied to a wide range of physical systems.

1.1 PHYSICAL CHARACTERISTICS OF SIMPLE HARMONIC OSCILLATORS

Observing the motion of a pendulum can tell us a great deal about the general characteristics of SHM. We could make such a pendulum by suspending an apple from the end of a length of string. When we draw the apple away from its *equilibrium position* and release it we see that the apple swings back towards the equilibrium position. It starts off from rest but steadily picks up speed. We notice that it *overshoots* the equilibrium position and does not stop until it reaches the other extreme of its motion. It then swings back toward the equilibrium position and eventually arrives back at its initial position. This pattern then repeats with the apple swinging backwards and forwards *periodically*. Gravity is the *restoring force* that attracts the apple back to its equilibrium position. It is the *inertia* of the mass that causes it to overshoot. The apple has kinetic energy because of its motion. We notice that its velocity is zero when its displacement from the equilibrium position is a maximum and so its kinetic energy is also zero at that point. The apple also has potential energy. When it moves away from the equilibrium position the apple's vertical height increases and it gains potential energy. When the apple passes through the equilibrium position its vertical displacement is zero and so all of its energy must be kinetic. Thus at the point of zero displacement the velocity has its maximum value. As the apple swings back and forth there is a continuous exchange between its potential and kinetic energies. These characteristics of the pendulum are common to all simple harmonic oscillators: (i) periodic motion; (ii) an equilibrium position; (iii) a restoring force that is directed towards this equilibrium position; (iv) inertia causing overshoot; and (v) a continuous flow of energy between potential and kinetic. Of course the oscillation of the apple steadily dies away due to the effects of dissipative forces such as air resistance, but we will delay the discussion of these effects until Chapter 2.

1.2 A MASS ON A SPRING

1.2.1 A mass on a horizontal spring

Our first example of a simple harmonic oscillator is a mass on a horizontal spring as shown in Figure 1.1. The mass is attached to one end of the spring while the other end is held fixed. The equilibrium position corresponds to the unstretched length of the spring and x is the displacement of the mass from the equilibrium position along the x-axis. We start with an idealised version of a real physical situation. It is idealised because the mass is assumed to move on a frictionless surface and the spring is assumed to be weightless. Furthermore because the motion is in the horizontal direction, no effects due to gravity are involved. In physics it is quite usual to start with a simplified version or model because real physical situations are normally complicated and hard to handle. The simplification makes the problem tractable so that an initial, idealised solution can be obtained. The complications, e.g. the effects of friction on the motion of the oscillator, are then added in turn and at each stage a modified and improved solution is obtained. This process invariably provides a great deal of physical understanding about the real system and about the relative importance of the added complications.

Figure 1.1 A simple harmonic oscillator consisting of a mass ***m*** on a horizontal spring.

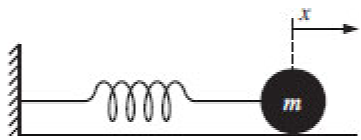
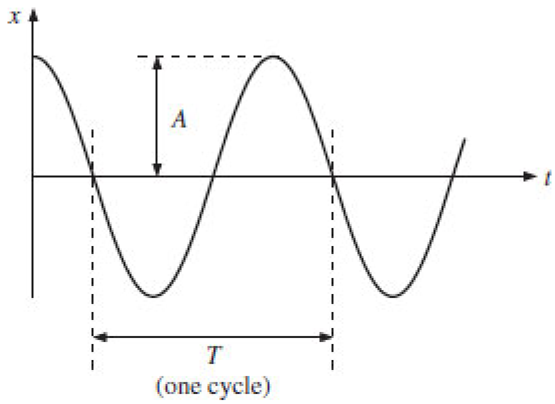


Figure 1.2 Variation of displacement x with time ***t*** for a mass undergoing SHM.



Experience tells us that if we pull the mass so as to extend the spring and then release it, the mass will move back and forth in a periodic way. If we plot the displacement x of the mass with respect to time t we obtain a curve like that shown in Figure 1.2. The *amplitude* of the oscillation is A, corresponding to the maximum excursion of the mass, and we note the *initial condition* that $x = A$ at time $t = 0$. The time for one complete cycle of oscillation is the period T. The frequency v is the number of cycles of oscillation per unit time. The relationship between period and frequency is

(1.1) $$\nu = \frac{1}{T}.$$

The units of frequency are hertz (Hz), where

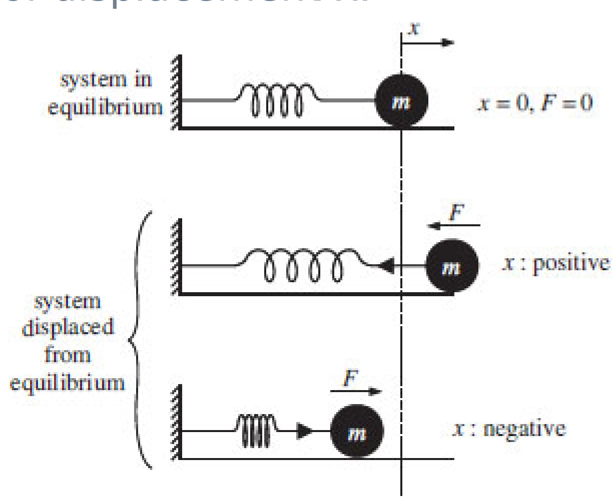
$$1\ \text{Hz} \equiv 1\ \text{cycle per second} \equiv 1\ \text{s}^{-1}.$$

For small displacements the force produced by the spring is described by Hooke's law which says that the strength of the force is proportional to the extension (or compression) of the spring, i.e. $F \propto x$ where x is the displacement of the mass. The constant of proportionality is the spring constant k which is defined as the force per unit displacement. When the spring is extended, i.e. x is positive, the force acts in the opposite direction to x to pull the mass back to the equilibrium position. Similarly when the spring is compressed, i.e. x is negative, the force again acts in the opposite direction to x to push the mass back to the equilibrium position. This situation is illustrated in Figure 1.3 which shows the direction of the force at various points of the oscillation. We can therefore write

(1.2) $F = -kx$

where the minus sign indicates that the force always acts in the opposite direction to the displacement. *All* simple harmonic oscillators have forces that act in this way: (i) the magnitude of the force is directly proportional to the displacement; and (ii) the force is always directed towards the equilibrium position.

Figure 1.3 The direction of the force acting on the mass *m* at various values of displacement *x*.



The system must also obey Newton's second law of motion which states that the force is equal to mass *m* times acceleration *a*, i.e. *F* = *ma*. We thus obtain the equation of motion of the mass

(1.3) $F = ma = -kx.$

Recalling that velocity *v* and acceleration *a* are, respectively, the first and second derivatives of displacement with respect to time, i.e.

(1.4) $a = \frac{dv}{dt} = \frac{d^2x}{dt^2},$

we can write Equation (1.3) in the form of the differential equation

(1.5) $m\frac{d^2x}{dt^2} = -kx$

or

(1.6) $\boxed{\frac{d^2x}{dt^2} = -\omega^2 x}$

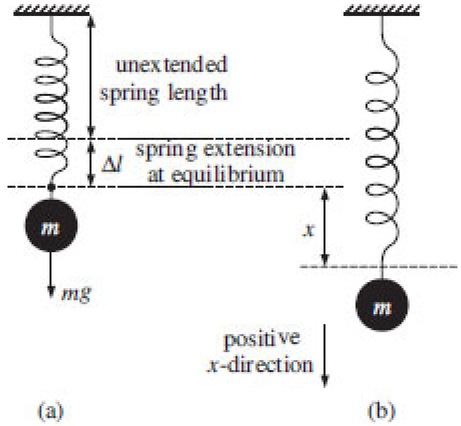
where

(1.7) $\omega^2 = \frac{k}{m}$

is a constant. Equation (1.6) is the equation of SHM and *all* simple harmonic oscillators have an equation of this form. It is a linear second-order differential equation; linear because each term is proportional to *x* or one of its derivatives and second order because the highest derivative occurring in it is second order. The reason for writing the constant as ω^2 will soon become apparent but we note that ω^2 is equal to the restoring force per unit displacement per unit mass.

1.2.2 A mass on a vertical spring

Figure 1.4 An oscillating mass on a vertical spring. (a) The mass at its equilibrium position. (b) The mass displaced by a distance ***x*** from its equilibrium position.



If we suspend a mass from a vertical spring, as shown in Figure 1.4, we have gravity also acting on the mass. When the mass is initially attached to the spring, the length of the spring increases by an amount Δt. Taking displacements in the downward direction as positive, the resultant force on the mass is equal to the gravitational force minus the force exerted upwards by the spring, i.e. the resultant force is given by $mg - k\Delta l$. The resultant force is equal to zero when the mass is at its equilibrium position. Hence

$$k\Delta l = mg.$$

When the mass is displaced downwards by an amount x, the resultant force is given by

$$F = m\frac{d^2x}{dt^2} = mg - k(\Delta l + x) = mg - k\Delta l - kx$$

i.e.

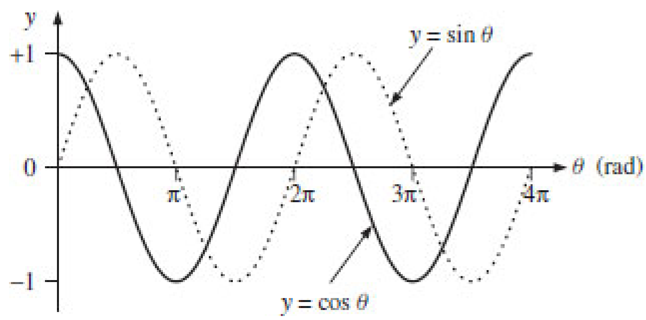
$$m\frac{d^2x}{dt^2} = -kx. \quad (1.8)$$

Perhaps not surprisingly, this result is identical to the equation of motion (1.5) of the horizontal spring: we simply need to measure displacements from the equilibrium position of the mass.

1.2.3 Displacement, velocity and acceleration in simple harmonic motion

To describe the harmonic oscillator, we need expressions for the displacement, velocity and acceleration as functions of time: $x(t)$, $v(t)$ and $a(t)$. These can be obtained by solving Equation (1.6) using standard mathematical methods. However, we will use our physical intuition to deduce them from the observed behaviour of a mass on a spring.

Figure 1.5 The functions $y = \cos\theta$ and $y = \sin\theta$ plotted over two complete cycles.



Observing the periodic motion shown in Figure 1.2, we look for a function $x(t)$ that also repeats periodically. Periodic functions that are familiar to us are sin θ and cos θ. These are reproduced in Figure 1.5 over two complete cycles. Both functions repeat every time the angle θ changes by 2π. We can notice that the two functions are identical except for a shift of π/2 along the θ axis. We also note the initial condition that the displacement x of the mass equals A at $t = 0$. Comparison of the actual motion with the mathematical functions in Figure 1.5 suggests the choice of a cosine function for $x(t)$. We write it as

$$x = A\cos\left(\frac{2\pi t}{T}\right) \quad (1.9)$$

which has the correct form in that $(2\pi t/T)$ is an angle (in radians) that goes from 0 to 2π as t goes from 0 to T, and so repeats with the correct period. Moreover x equals A at $t = 0$ which matches the initial condition. We also require that $x = A\cos(2\pi t/T)$ is a solution to our differential equation (1.6). We define

(1.10) $$\omega = \frac{2\pi}{T}$$

where ω is the *angular frequency* of the oscillator, with units of rad s^{-1} to obtain

(1.11) $$x = A\cos\omega t.$$

Then

(1.12) $$\frac{dx}{dt} = v = -\omega A\sin\omega t,$$

and

(1.13) $$\frac{d^2x}{dt^2} = a = -\omega^2 A\cos\omega t = -\omega^2 x.$$

So, the function $x = A\cos\omega t$ is a solution of Equation (1.6) and correctly describes the physical situation. The reason for writing the constant as ω^2 in Equation (1.6) is now apparent: the constant is equal to the square of the angular frequency of oscillation. We have also obtained expressions for the velocity v and acceleration a of the mass as functions of time. All three functions are plotted in Figure 1.6. Since they relate to different physical quantities, namely displacement, velocity and acceleration, they are plotted on separate sets of axes, although the time axes are aligned with respect to each other.

Figure 1.6 (a) The displacement x, (b) the velocity v and (c) the acceleration a of a mass undergoing SHM as a function of time t. The time axes of the three graphs are aligned.

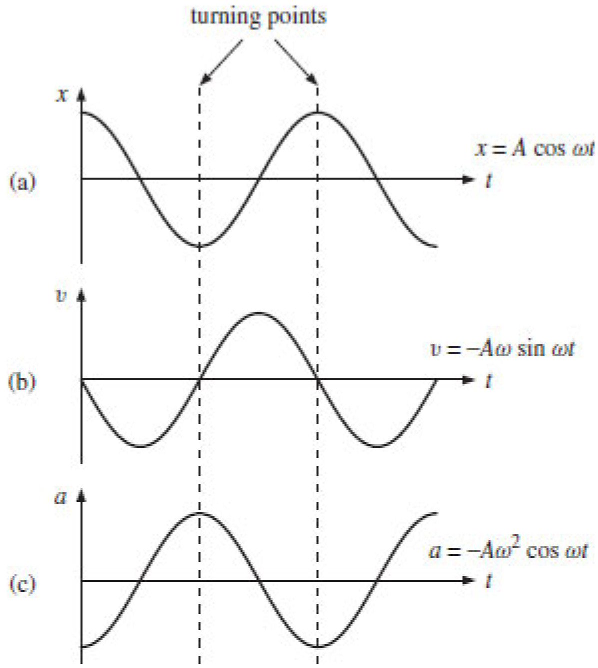


Figure 1.6 shows that the behaviour of the three functions (1.11)−(1.13) agree with our observations. For example, when the displacement of the mass is greatest, which occurs at the *turning points* of the motion ($x = \pm A$), the velocity is zero. However, the velocity is at a maximum when the mass passes through its equilibrium position, i.e. $x = 0$. Looked at in a different way, we can see that the maximum in the velocity curve occurs before the maximum in the displacement curve by one quarter of a period which corresponds to an angle of $\pi/2$. We can understand at which points the maxima and minima of the acceleration occur by recalling that acceleration is directly proportional to the force. The force is maximum at the turning points of the motion but is of opposite sign to the displacement. The acceleration does indeed follow this same pattern, as is readily seen in Figure 1.6.

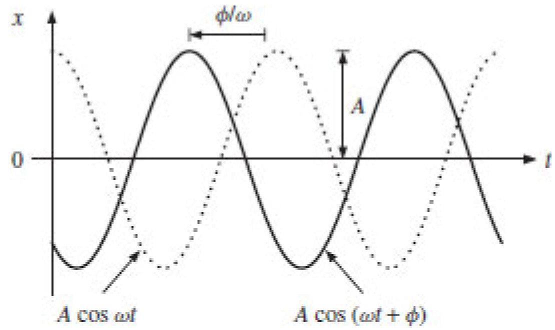
1.2.4 General solutions for simple harmonic motion and the phase angle ϕ

In the example above, we had the particular situation where the mass was released from rest with an initial displacement *A*, i.e. *x* equals A at $t = 0$. For the more general case, the motion of the oscillator will give rise to a displacement curve like that shown by the solid curve in Figure 1.7, where the displacement and velocity of the mass have arbitrary values at $t = 0$. This solid curve looks like the cosine function $x = A \cos \omega t$, that is shown by the dotted curve, but it is displaced horizontally to the left of it by a time interval $\phi/\omega = \phi T/2\pi$. The solid curve is described by

(1.14) $$x = A\cos(\omega t + \phi)$$

where again *A* is the amplitude of the oscillation and ϕ is called the *phase angle* which has units of radians. [Note that changing ωt to $(\omega t - \phi)$ would shift the curve to the right in Figure 1.7.] Equation (1.14) is also a solution of the equation of motion of the mass, Equation (1.6), as the reader can readily verify. In fact Equation (1.14) is the *general solution* of Equation (1.6). We can state here a property of second-order differential equations that they always contain two arbitrary constants. In this case *A* and ϕ are the two constants which are determined from the initial conditions, i.e. from the position and velocity of the mass at time $t = 0$.

Figure 1.7 General solution for displacement *x* in SHM showing the phase angle ϕ, where $x = A\cos(\omega t + \phi)$.



We can cast the general solution, Equation (1.14), in the alternative form

(1.15) $x = a\cos\omega t + b\sin\omega t,$

where *a* and *b* are now the two constants. Equations (1.14) and (1.15) are entirely equivalent as we can show in the following way. Since

(1.16) $A\cos(\omega t + \phi) = A\cos\omega t\cos\phi - A\sin\omega t\sin\phi$

and cos ϕ and sin ϕ have constant values, we can rewrite the right-hand side of this equation as

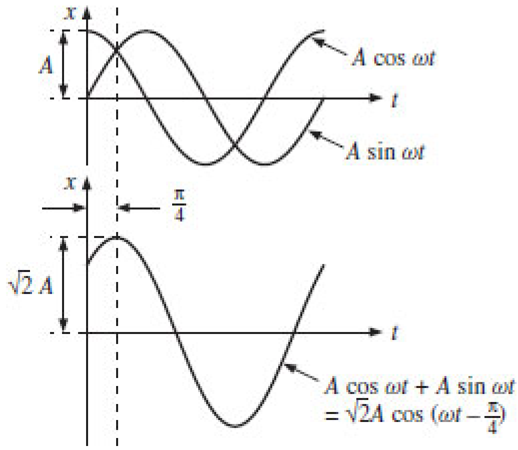
$a\cos\omega t + b\sin\omega t,$

where

(1.17) $a = A\cos\phi$ and $b = -A\sin\phi$.

We see that if we add sine and cosine curves of the *same* angular frequency ω, we obtain another cosine (or corresponding sine curve) of angular frequency ω. This is illustrated in Figure 1.8 where we plot $A\cos\omega t$ and $A\sin\omega t$, and also (A cos ωt + A sin ωt) which is equal to $\sqrt{2}A\cos(\omega t - \pi/4)$. As the motion of a simple harmonic oscillator is described by sines and cosines it is called harmonic and because there is only a single frequency involved, it is called simple harmonic.

Figure 1.8 The addition of sine and cosine curves with the same angular frequency ω. The resultant curve also has angular frequency ω.



There is an important difference between the constants A and ϕ in the general solution for SHM given in Equation (1.14) and the angular frequency ω. The constants are determined by the initial conditions of the motion. However, the angular frequency of oscillation ω is determined only by the properties of the oscillator: the oscillator has a *natural frequency of oscillation* that is independent of the way in which we start the motion. This is reflected in the fact that the SHM equation, Equation (1.6), already contains ω which therefore has nothing to do with any particular solutions of the equation. This has important practical applications. It means, for example, that the period of a pendulum clock is independent of the amplitude of the pendulum so that it keeps time to a high degree of accuracy.[1] It means that the pitch of a note from a piano does not depend on how hard you strike the keys. For the example of the mass on a spring, $\omega = \sqrt{k/m}$. This expression tells us that the angular frequency becomes lower as the mass increases and becomes higher as the spring constant increases.

Worked example

In the example of a mass on a horizontal spring (cf. Figure 1.1) *m* has a value of 0.80 kg and the spring constant *k* is 180 N m^{-1}. At time *t* = 0 the mass is observed to be 0.04 m further from the wall than the equilibrium position and is moving away from the wall with a velocity of 0.50 m s^{-1}. Obtain an expression for the displacement of the mass in the form $x = A(\cos\omega t + \phi)$, obtaining numerical values for A, ω and ϕ.

Solution

The angular frequency ω depends only on the oscillator parameters *k* and *m*, and not on the initial conditions. Substituting their values gives

$$\omega = \sqrt{k/m} = 15.0 \text{ rad s}^{-1}$$

To find the amplitude *A*: From $x = A\cos(\omega t + \phi)$ we obtain

$$v = -A\omega \sin(\omega t + \phi).$$

Substituting the initial values (i.e. at time $t = 0$), of *x* and *v* into these equations gives

$$0.04 = A\cos\phi, \quad 0.50 = -15A\sin\phi.$$

From $\cos^2\phi + \sin^2\phi = 1$, we obtain A = 0.052 m.

To find the phase angle ϕ: Substituting the value for *A* leads to two equations for ϕ:

$$\cos\phi = 0.04/0.052, \qquad \text{giving } \phi = 39.8^\circ \text{ or } 320^\circ,$$

$$\sin\phi = -0.50/(15 \times 0.052), \qquad \text{giving } \phi = -39.8^\circ \text{ or } 320^\circ.$$

Since ϕ must satisfy both equations, it must have the value $\phi = 320°$. The angular frequency ω is given in rad s^{-1}. To convert ϕ to radians:

$$\phi = (\pi/180) \times 320 \text{ rad} = 5.59 \text{ rad. Hence, } x = 0.052\cos(15t + 5.59) \text{ m.}$$