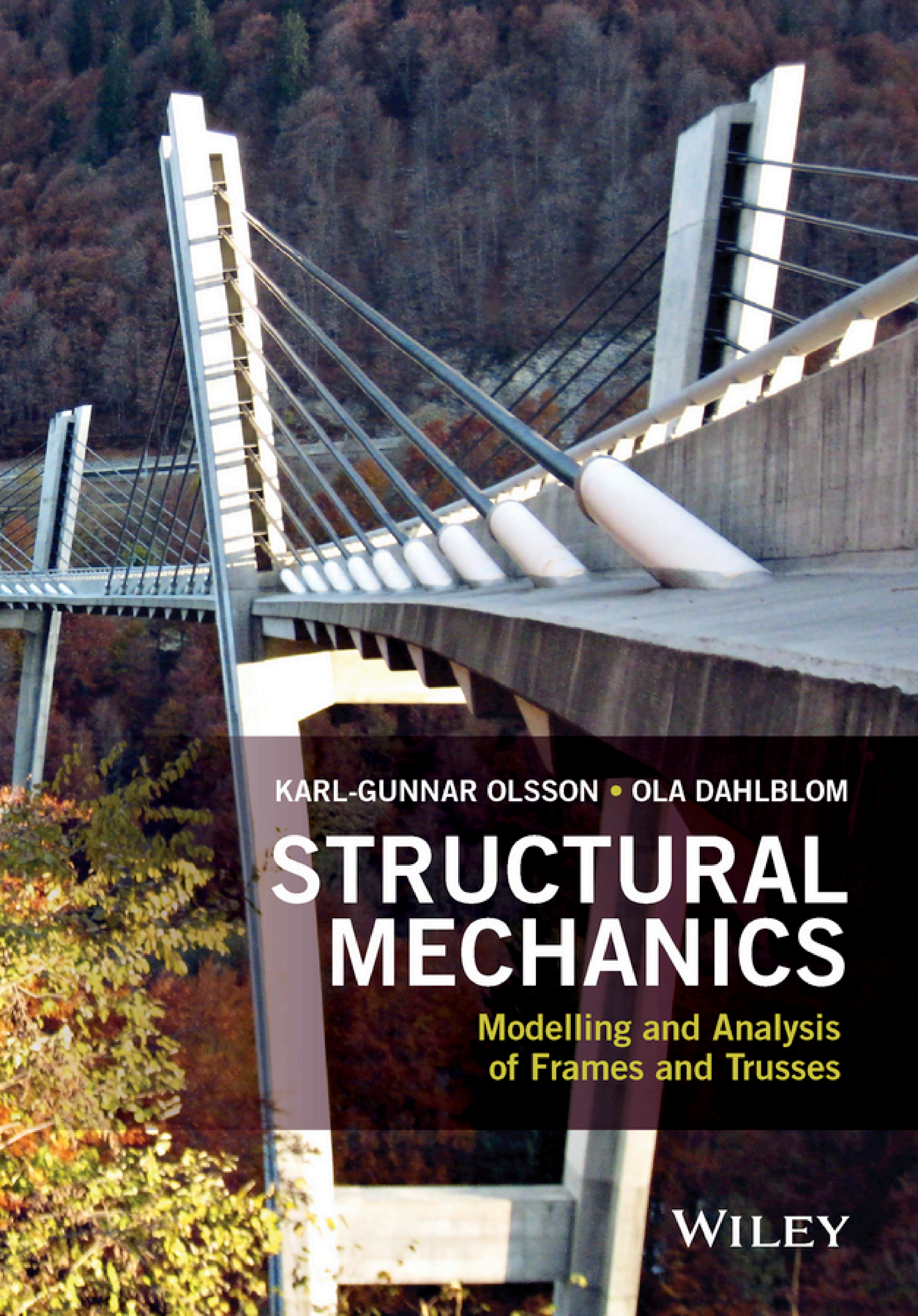

KARL-GUNNAR OLSSON • OLA DAHLBLOM

STRUCTURAL MECHANICS

Modelling and Analysis of Frames and Trusses

WILEY

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MODELLING AND ANALYSIS OF FRAMES AND TRUSSES

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Preface

The autumn sun shines on Sunnibergbrücke at Klosters in the canton of Graubünden in south-western Switzerland. On the cover picture one can sense how the bridge elegantly migrates through the landscape. The steel and concrete structure and the architecture merge into one of the most elegant buildings of our time. The engineer who designed the bridge is named Christian Menn. It is late in October 2009, and a group of Swedish students sketch, photograph and enthusiastically discuss the shape and the structural behaviour of the bridge. In a week they will start a course in structural mechanics.

Structural mechanics is the branch of physics that describes how different materials, which have been shaped and joined together to structures, carry their loads. Knowledge on the modes of action of these structures can be used in different contexts and for different purposes. The Roman architect and engineer Vitruvius, who lived during the first century BC summarises in the work *De architectura libri decem* ('Ten books on architecture') the art of building with the three classical notions of *firmitas*, *utilitas* and *venustas* (strength, functionality and beauty). Engineering of our time has basically the same goal. It is about utilising the knowledge and practices of our time in a creative process where sustainable and efficient, functional and expressive buildings are designed.

At an early design stage a structural engineer needs to be trained to see how to efficiently use material and shape to provide the construction with stability, stiffness and strength. Using simple models, structural behaviour can be evaluated and cross-section sizes estimated. As the design develops the need for precision of the analyses increases. In all this, the ability to formulate computational models and to carry out simulations is of crucial importance.

A useful computational model should be simple enough to be easily manageable and, simultaneously, sufficiently complex to provide an adequate accuracy. In recent years, the finite element method has become the dominant method for formulating computational models and conducting analyses. The FE method is based on expressing forces and deformations as discrete entities in a chosen and representative set of degrees of freedom. Between the degrees of freedom simple bodies (elements) are placed and together they constitute the structure to be modelled. Each element may describe a unique mode of action and can be given a specific geometry. In all this, FEM provides opportunities for both accurate analyses of structures with complex geometry and material behaviour, and for quick estimates in early design stages.

Here, we present a new textbook in structural mechanics, dealing with the modelling and analysis of trusses and frames. The textbook is based on the finite element method. Gradually, an understanding of basic elements of structural mechanics – springs, bars, beams, foundations and so on is built up. Methods for assembling them into complex load-bearing structures are presented, and tools for analysis and simulation are provided. The book has been limited to treating trusses and frames in two and three dimensions. To demonstrate the generality of the methodology the book also has a chapter, ‘Flows in Networks’, that addresses other areas of applied mechanics, including thermal conduction and electrical flow.

The textbook supports three kinds of learning outcome:

- *Knowledge of basic theory of structural mechanics*. The textbook has a structure that highlights the theory as a whole. Different modes of action in structural mechanics are described in a common format where basic concepts and relationships recur at different scale levels. One aim is to highlight the mechanisms that determine how structures carry their loads and how we by this knowledge can manipulate the distribution of internal forces as well as patterns of deformations.
- *Skills in modelling and analysis of structures*. Being able to describe a structure by a mathematical model and perform computations is one of the most important engineering skills. The matrix-based presentation of the textbook practices a computation methodology that is general and can be applied for phenomena and geometries of structural mechanics as well as for simulations in a variety of engineering areas far beyond the textbook limitations. Through exercises and with support from the computer program Matlab/CALFEM students in a course formulate about 30 computer algorithms of their own, each with increasing complexity.
- *Ability to evaluate and optimise designs proposed*. Having an eye trained for patterns of forces and deformations helps to evaluate and improve the efficiency of structural designs. This facilitates modification of the design of a structure in the desired direction, thus creating an efficient structural behaviour, for example by reducing bending in the favour of axial only forces – compression and tension.

The textbook is intended for engineering students at the bachelor level. The presentation assumes knowledge of calculus in one variable, linear algebra, classical mechanics and basic solid/structural mechanics. Chapters 1–5 are a unit and should be read in the order they appear, while Chapters 6–10 are independent of each other and can be read in any order. For a limited course, we recommend primarily Chapters 1–6.

The Division of Structural Mechanics at Lund University has a long tradition in the development of teaching materials in structural mechanics and the finite element method. A key person behind this development is Hans Petersson who came to the division as a professor in 1977. Within a few years, a group of young Ph.D. students and teachers gathered around Hans, taking note of his knowledge and absorbed his enthusiasm about teaching and its tools. We were two of them. Earlier, the framework of the computer program CALFEM (Computer Aided Learning of the Finite Element Method) was developed, and based on his concept the textbook ‘Konstruktionsberäkningar med dator’ (Design calculations using a computer) was written with Sven Thelandersson as author. In this spirit, the division has continued to develop teaching materials, and approaches. In more than 30 years time, both ideas and collaborators spread. CALFEM is today a toolbox to the computer program Matlab and is used worldwide. In Sweden, collaboration between Lund University, Chalmers and KTH Royal Institute of Technology has been established, and from the site www.structarch.org, CALFEM as well as other software for structural mechanics analysis and conceptual design can be downloaded free of charge.

The contents of this textbook have been developed over many years and there are many students and colleagues at Lund University, Chalmers and Linnæus University, who contributed with ideas, suggestions, corrections and translations during the creation of the book. We would particularly like to mention Professor Per-Erik Austrell, Dr. Henrik Danielsson, Dr. Susanne Heyden and Professor Kent Persson at Structural Mechanics in Lund, Dr. Mats Ander and Dr. Peter Möller at Applied Mechanics at Chalmers and Ms. Louise Blyberg and Professor Anders Olsson at Linnæus University in Växjö. Professor Emeritus Bengt Åkesson at Chalmers has with great precision and sharpness examined facts of the manuscript and given us reason to examine and modify the conceptual choices and formulations. Dr. Samar Malek has thoroughly proofread the English version of the text. Mr. Bo Zadig at Structural Mechanics in Lund has skilfully drawn the figures. Sincere thanks to all of you for your commitment and wise observations. And to Professor Göran Sandberg who with his character, his knowledge and in his role as head of the department has built and continues to build a creative environment for the teaching and development of teaching concepts and tools. We want to thank people at John Wiley & Sons and their partners for cooperation and guidance. In particular we are grateful to Eric Willner, Anne Hunt, Clive Lawson and Lincy Priya.

The textbook is also available in Swedish, with the reverse order of authors.

Karl-Gunnar Olsson and Ola Dahlblom
Gothenburg and Lund in October 2015

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Matrix Algebra

The method used in this textbook to formulate computational models is characterised by the use of matrices. The different quantities – load, section force, stiffness and displacement – are separated and gathered into groups of numbers. All load values are gathered in a load matrix and all stiffnesses in a stiffness matrix. This is one of the primary strengths of the method. With a matrix formulation, the formulae describing the relations between quantities are compact and easy to view. Physical mechanisms and underlying principles become clear. We begin with a short summary of the matrix algebra and the notations that are used.

1.1 Definitions

A matrix consists of a set of *matrix elements* ordered in *rows* and *columns*. If the matrix consists of only one column it is referred to as a *column matrix* and if it has only one row it is referred to as a *row matrix*. Such matrices are *one-dimensional* and may also be referred to as *vectors*. A vector is denoted by a lower case letter set in bold:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{1.1}$$

where a_1, a_2 and a_3 are the components of the vector. A *two-dimensional* matrix is denoted by a capital letter set in bold:

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ A_{41} & A_{42} & A_{43} \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \tag{1.2}$$

where A_{11}, A_{12} and so on are elements of the matrix $\mathbf{A}$. An arbitrary component of a matrix is denoted A_{ij}, where the first index refers to the row number and the second index to the column number. The matrix $\mathbf{A}$ in (1.2) has the *dimensions* 4×3 and the matrix $\mathbf{B}$ has the dimensions 3×3.

Structural Mechanics: Modelling and Analysis of Frames and Trusses, First Edition.
Karl-Gunnar Olsson and Ola Dahlblom.

Since the number of rows and columns in **B** are equal, it is a *square matrix*. If it is only the *diagonal elements* B_{ii} that are different from 0, the matrix is a *diagonal matrix*. A diagonal matrix where all the diagonal elements are equal to 1 is an *identity matrix* and is usually denoted **I**. The *transposed matrix* $\mathbf{A}^T$ of a matrix **A** is formed by letting the rows of **A** become columns of $\mathbf{A}^T$, that is the *transpose* of **A** in (1.2) is

$$\mathbf{A}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} & A_{41} \\ A_{12} & A_{22} & A_{32} & A_{42} \\ A_{13} & A_{23} & A_{33} & A_{43} \end{bmatrix} \tag{1.3}$$

A matrix **A** is *symmetric* if $\mathbf{A} = \mathbf{A}^T$. Only square matrices can be symmetric. A matrix with all elements equal to 0 is referred to as a *zero matrix* and is usually denoted **0**.

1.2 Addition and Subtraction

Matrices of equal dimensions can be added and subtracted. The result is a new matrix of the same dimensions, where each element is the sum of or the difference between the corresponding elements of the two matrices. If

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \tag{1.4}$$

the sum of **A** and **B** is given by

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{1.5}$$

where

$$\mathbf{C} = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} & A_{13}+B_{13} \\ A_{21}+B_{21} & A_{22}+B_{22} & A_{23}+B_{23} \\ A_{31}+B_{31} & A_{32}+B_{32} & A_{33}+B_{33} \end{bmatrix} \tag{1.6}$$

and the difference between **A** and **B** is given by

$$\mathbf{D} = \mathbf{A} - \mathbf{B} \tag{1.7}$$

where

$$\mathbf{D} = \begin{bmatrix} A_{11}-B_{11} & A_{12}-B_{12} & A_{13}-B_{13} \\ A_{21}-B_{21} & A_{22}-B_{22} & A_{23}-B_{23} \\ A_{31}-B_{31} & A_{32}-B_{32} & A_{33}-B_{33} \end{bmatrix} \tag{1.8}$$

1.3 Multiplication

Multiplying a matrix **A** with a scalar c results in a matrix with the same dimensions as **A** and where each element is the corresponding element of **A** multiplied by c, that is

$$c\mathbf{A} = \begin{bmatrix} cA_{11} & cA_{12} & cA_{13} \\ cA_{21} & cA_{22} & cA_{23} \\ cA_{31} & cA_{32} & cA_{33} \end{bmatrix} \tag{1.9}$$

Multiplication between two matrices

$$\mathbf{C} = \mathbf{AB} \tag{1.10}$$

can be performed only if the number of columns in **A** equals the number of rows in **B**. The element C_{ij} is then computed according to

$$C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} \tag{1.11}$$

For

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{1.12}$$

the product of the matrices, $\mathbf{C} = \mathbf{AB}$, is obtained from

$$\mathbf{C} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} \tag{1.13}$$

In general,

$$\mathbf{BA} \neq \mathbf{AB} \tag{1.14}$$

1.4 Determinant

For every quadratic matrix **A** ($n \times n$), it is possible to compute a scalar value called a *determinant*. For $n = 1$,

$$\det \mathbf{A} = A_{11} \tag{1.15}$$

For $n > 1$, the determinant $\det \mathbf{A}$ is computed according to the expression

$$\det \mathbf{A} = \sum_{k=1}^{n} (-1)^{i+k} A_{ik} \det M_{ik} \tag{1.16}$$

where i is an arbitrary row number and $\det M_{ik}$ is the determinant of the matrix obtained when the ith row and the kth column is deleted from the matrix **A**. For $n = 2$, this results in

$$\det \mathbf{A} = A_{11}A_{22} - A_{12}A_{21} \tag{1.17}$$

and for $n = 3$

$$\det \mathbf{A} = A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} - A_{13}A_{22}A_{31} \tag{1.18}$$

1.5 Inverse Matrix

The quadratic matrix **A** is *invertible* if there exists a matrix $\mathbf{A}^{-1}$ such that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \tag{1.19}$$

The matrix $\mathbf{A}^{-1}$ is then the *inverse* of $\mathbf{A}$. For the inverse $\mathbf{A}^{-1}$ to exist, it is necessary that $\det \mathbf{A} \neq 0$. If

$$\mathbf{A}^{-1} = \mathbf{A}^T \tag{1.20}$$

the matrix $\mathbf{A}$ is *orthogonal* and then

$$\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{I} \tag{1.21}$$

1.6 Counting Rules

The following counting rules apply to matrices (under the condition that the dimensions of the matrices included are such that the operations are defined).

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{1.22}$$

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} \tag{1.23}$$

$$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T \tag{1.24}$$

$$(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T \tag{1.25}$$

$$\mathbf{I}\mathbf{A} = \mathbf{A} \tag{1.26}$$

$$c(\mathbf{A}\mathbf{B}) = (c\mathbf{A})\mathbf{B} = \mathbf{A}(c\mathbf{B}) \tag{1.27}$$

$$(c + d)\mathbf{A} = c\mathbf{A} + d\mathbf{A} \tag{1.28}$$

$$c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B} \tag{1.29}$$

$$(\mathbf{A}\mathbf{B})\mathbf{C} = \mathbf{A}(\mathbf{B}\mathbf{C}) \tag{1.30}$$

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{A}\mathbf{C} + \mathbf{B}\mathbf{C} \tag{1.31}$$

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{A}\mathbf{B} + \mathbf{A}\mathbf{C} \tag{1.32}$$

$$\det \mathbf{A}\mathbf{B} = \det \mathbf{A} \det \mathbf{B} \tag{1.33}$$

$$\det \mathbf{A}^{-1} = 1/\det \mathbf{A} \tag{1.34}$$

$$\det c\mathbf{A} = c^n \det \mathbf{A} \tag{1.35}$$

$$(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1} \tag{1.36}$$

$$(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \tag{1.37}$$

1.7 Systems of Equations

A *linear system of equations* with n equations and p unknowns can be written in matrix form as

$$\mathbf{K}\,\mathbf{a} = \mathbf{f} \tag{1.38}$$

where $\mathbf{K}$ has the dimensions $n \times p$, $\mathbf{a}$ the dimensions $p \times 1$ and $\mathbf{f}$ the dimensions $n \times 1$. Usually, the coefficients in $\mathbf{K}$ are known, while the coefficients in $\mathbf{a}$ and $\mathbf{f}$ can be known as well as unknown. For the case when all the components of **a** are unknown and all the components of **f** are known, there are three types of systems of equations:

- $n = p$, the number of equations equals the number of unknowns. The matrix **K** is quadratic. Depending on the contents of **K** and **f**, four different characteristic cases can be recognised. These are often indications of different states or behaviours that may be important to notice: If $\det \mathbf{K} \neq 0$, there is a *unique solution*.
  - For $\mathbf{f} = \mathbf{0}$, this solution is the trivial one, $\mathbf{a} = \mathbf{0}$.
  - For $\mathbf{f} \neq \mathbf{0}$, there is a unique solution, $\mathbf{a} \neq \mathbf{0}$. In general, this is an indication of a functioning physical model.

  If $\det \mathbf{K} = 0$, there is no unique solution. This may be an indication of an, in some way, unstable physical model.
  - For $\mathbf{f} = 0$, there are infinitely many solutions. This is the case for eigenvalue problems, which, for example, can be a method to gain knowledge about unstable states of the model.
  - For $\mathbf{f} \neq 0$, there is either none or infinitely many solutions; there may be elements missing in the model or the set of boundary conditions may be incomplete.
- $n < p$, the number of equations is less than the number of unknowns. The system is underdetermined. There are infinitely many solutions.
- $n > p$, the number of equations exceeds the number of unknowns. The system is overdetermined. In general, there is no solution.

In the following symmetric matrices, **K** and **A** are considered which are common in the forthcoming applications.

1.7.1 Systems of Equations with Only Unknown Components in the Vector a

For the case when $\det \mathbf{K} \neq 0$ and $\mathbf{f} \neq \mathbf{0}$, the unknowns in the vector **a** can be determined by Gaussian elimination. This is shown in the following example.

Example 1.1 Solving a system of equations with only unknown components in the vector a

We are looking for a solution to the system of equations

$$\begin{bmatrix} 8 & -4 & -2 \\ -4 & 10 & -4 \\ -2 & -4 & 10 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 18 \\ 6 \end{bmatrix} \tag{1}$$

The unknowns are determined by Gaussian elimination. In this procedure, all elements different from 0 are eliminated below the diagonal: let the first row remain unchanged. From row 2 we subtract row 1 multiplied by the quotient $K_{21}/K_{11} = -4/8 = -0.5$. From row 3 we subtract row 1 multiplied by the quotient $K_{31}/K_{11} = -2/8 = -0.25$. In this way, we obtain

$$\begin{bmatrix} 8 & -4 & -2 \\ 0 & 8 & -5 \\ 0 & -5 & 9.5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 14 \\ 4 \end{bmatrix} \tag{2}$$

In the next step, we let the rows 1 and 2 remain. From row 3 we subtract row 2 multiplied by the quotient $K_{32}/K_{22} = -5/8 = -0.625$. We have triangularised the coefficient matrix **K** and obtain

$$\begin{bmatrix} 8 & -4 & -2 \\ 0 & 8 & -5 \\ 0 & 0 & 6.375 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 14 \\ 12.75 \end{bmatrix} \tag{3}$$

With the system of equations in this form, we can determine a_3, a_2 and a_1 by back-substitution

$$a_3 = \frac{12.75}{6.375} = 2; \quad a_2 = \frac{14 - (-5)a_3}{8} = 3;$$

$$a_1 = \frac{-8 - (-4)a_2 - (-2)a_3}{8} = 1 \tag{4}$$

and with that, we have the solution

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \tag{5}$$

To check the results, we can substitute the solution into the original system of equations and carry out the matrix multiplication

$$\begin{bmatrix} 8 & -4 & -2 \\ -4 & 10 & -4 \\ -2 & -4 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad \text{which gives} \quad \begin{bmatrix} -8 \\ 18 \\ 6 \end{bmatrix} \tag{6}$$

This is equal to the original right-hand side of the system of equations, that is the solution found is correct.

1.7.2 Systems of Equations with Known and Unknown Components in the Vector **a**

The systems of equations that we consider, in general, has a square matrix **K**, initially with $\det \mathbf{K} = 0$, and a vector $\mathbf{f} \neq \mathbf{0}$. Moreover, it is usually the case that some components of **a** are known and the corresponding components of **f** are unknown. One systematic way to solve such a system of equations begins with a *partition* of the matrices, which means that they are divided into *submatrices*

$$\mathbf{K} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \tilde{\mathbf{K}} \end{bmatrix}; \quad \mathbf{a} = \begin{bmatrix} \mathbf{g} \\ \tilde{\mathbf{a}} \end{bmatrix}; \quad \mathbf{f} = \begin{bmatrix} \mathbf{r} \\ \tilde{\mathbf{f}} \end{bmatrix} \tag{1.39}$$

where the matrices $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \tilde{\mathbf{K}}, \mathbf{g}$ and $\tilde{\mathbf{f}}$ contain known quantities, while $\tilde{\mathbf{a}}$ and $\mathbf{r}$ are unknown. With use of these submatrices, the system of equations (1.38) can be expressed as

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \tilde{\mathbf{K}} \end{bmatrix} \begin{bmatrix} \mathbf{g} \\ \tilde{\mathbf{a}} \end{bmatrix} = \begin{bmatrix} \mathbf{r} \\ \tilde{\mathbf{f}} \end{bmatrix} \tag{1.40}$$

The system of equations can be divided into two parts and then be written as

$$\mathbf{A}_1\mathbf{g} + \mathbf{A}_2\tilde{\mathbf{a}} = \mathbf{r} \tag{1.41}$$

$$\mathbf{A}_3\mathbf{g} + \tilde{\mathbf{K}}\tilde{\mathbf{a}} = \tilde{\mathbf{f}} \tag{1.42}$$

or

$$\tilde{\mathbf{K}}\tilde{\mathbf{a}} = \tilde{\mathbf{f}} - \mathbf{A}_3\mathbf{g} \tag{1.43}$$

$$\mathbf{r} = \mathbf{A}_1\mathbf{g} + \mathbf{A}_2\tilde{\mathbf{a}} \tag{1.44}$$

where the right-hand side of the equation (1.43) consists of known quantities. The purpose of the partition of the system of equations is to, within the original system of equations, find a sub-system with $\det \tilde{\mathbf{K}} \neq 0$, that is a system with a unique solution. The unknowns in $\tilde{\mathbf{a}}$ can then be computed from (1.43). One way to perform this computation is to use Gaussian elimination. Once $\tilde{\mathbf{a}}$ has been determined, $\mathbf{r}$ can be computed from (1.44).

Example 1.2 Solving a system of equations with both known and unknown components in the vector a

In the system of equations

$$\left[\begin{array}{cccc|cc} 20 & 0 & 0 & 0 & -20 & 0 \\ 0 & 15 & 0 & -15 & 0 & 0 \\ 0 & 0 & 16 & 12 & -16 & -12 \\ 0 & -15 & 12 & 24 & -12 & -9 \\ \hline -20 & 0 & -16 & -12 & 36 & 12 \\ 0 & 0 & -12 & -9 & 12 & 9 \end{array}\right] \left[\begin{array}{c} 0 \\ 0 \\ -3 \\ 0 \\ \hline a_5 \\ a_6 \end{array}\right] = \left[\begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \\ \hline 0 \\ -15 \end{array}\right] \tag{1}$$

the vector $\mathbf{a}$ has known and unknown components. The solution can then be systematised using partitioning (1.40). The auxiliary lines show this partition. The system of equations is partitioned into two parts according to (1.41) and (1.42):

$$\begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 15 & 0 & -15 \\ 0 & 0 & 16 & 12 \\ 0 & -15 & 12 & 24 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} -20 & 0 \\ 0 & 0 \\ -16 & -12 \\ -12 & -9 \end{bmatrix} \begin{bmatrix} a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} -20 & 0 & -16 & -12 \\ 0 & 0 & -12 & -9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} 36 & 12 \\ 12 & 9 \end{bmatrix} \begin{bmatrix} a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -15 \end{bmatrix} \tag{3}$$

In the lower system of equations, there are two equations and two unknowns. If the known terms of the system are gathered on the right-hand side of the equal sign, cf. (1.43), we obtain

$$\begin{bmatrix} 36 & 12 \\ 12 & 9 \end{bmatrix} \begin{bmatrix} a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -15 \end{bmatrix} - \begin{bmatrix} -20 & 0 & -16 & -12 \\ 0 & 0 & -12 & -9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \end{bmatrix} \tag{4}$$

or

$$\begin{bmatrix} 36 & 12 \\ 12 & 9 \end{bmatrix} \begin{bmatrix} a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} -48 \\ -51 \end{bmatrix} \tag{5}$$

From this system of equations, the unknown elements can be determined by Gaussian elimination: the first row remains unchanged. From row 2 we subtract row 1 multiplied by the quotient $K_{21}/K_{11} = 12/36 = 0.33333$. In this way, we obtain

$$\begin{bmatrix} 36 & 12 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} -48 \\ -35 \end{bmatrix} \tag{6}$$

and the unknown a_5 and a_6 can be determined by back-substitution

$$a_6 = \frac{-35}{5} = -7; \quad a_5 = \frac{-48 - 12a_6}{36} = 1 \tag{7}$$

$$\begin{bmatrix} a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \end{bmatrix} \tag{8}$$

With a_5 and a_6 being known, the unknown coefficients in $\mathbf{f}$ can be determined using the upper system of equations obtained from the partition, cf. (1.44),

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 15 & 0 & -15 \\ 0 & 0 & 16 & 12 \\ 0 & -15 & 12 & 24 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} -20 & 0 \\ 0 & 0 \\ -16 & -12 \\ -12 & -9 \end{bmatrix} \begin{bmatrix} 1 \\ -7 \end{bmatrix} = \begin{bmatrix} -20 \\ 0 \\ 20 \\ 15 \end{bmatrix} \tag{9}$$

and with that, all the unknowns are determined.

1.7.3 Eigenvalue Problems

At times it is of interest to study the case when $\det \mathbf{K} = 0$ and $\mathbf{f} = \mathbf{0}$. Mainly, two different types of problems appear. A system of equations in the form

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{a} = \mathbf{0} \tag{1.45}$$

is referred to as an *eigenvalue problem* or sometimes *standard eigenvalue problem*. For a solution to exist, it is required that

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{1.46}$$

A system of equations in the form

$$(\mathbf{A} - \lambda\mathbf{B})\mathbf{a} = \mathbf{0} \tag{1.47}$$

is referred to as a *generalised eigenvalue problem* and for a solution to exist it is required that

$$\det(\mathbf{A} - \lambda\mathbf{B}) = 0 \tag{1.48}$$

Solving an eigenvalue problem means that the values of λ, which fulfil Equations (1.46) and (1.48) are determined, that is the eigenvalues λ_i are computed. The number of eigenvalues

λ_i is equal to the number of unknowns in the system of equations. Two or more eigenvalues may coincide. A symmetric matrix **K** with real elements has only real eigenvalues. For each eigenvalue λ_i there is an eigenvector $\mathbf{a}_i$. The unknowns in the eigenvector $\mathbf{a}_i$ cannot be uniquely determined, but their relative magnitude can be computed.

If the product of two vectors $\mathbf{b}^T\mathbf{c} = 0$, then the vectors **b** and **c** are orthogonal. For eigenvectors, we have $\mathbf{a}_i^T\mathbf{a}_j = 0$ for $i \neq j$, that is any two eigenvectors are always orthogonal.

The following example shows how an eigenvalue problem is solved:

Example 1.3 Solving an eigenvalue problem

We want to find a solution to the eigenvalue problem

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{a} = \mathbf{0} \tag{1}$$

where

$$\mathbf{A} = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix} \tag{2}$$

The determinant of $(\mathbf{A} - \lambda\mathbf{I})$ can be computed as

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{bmatrix} 5-\lambda & -2 \\ -2 & 8-\lambda \end{bmatrix} = (5-\lambda)(8-\lambda) - 4 = \lambda^2 - 13\lambda + 36 \tag{3}$$

When this expression is set to zero, the equation

$$\lambda^2 - 13\lambda + 36 = 0 \tag{4}$$

is obtained. The solutions to this equation are the eigenvalues

$$\lambda_1 = 4; \quad \lambda_2 = 9 \tag{5}$$

By substituting the computed eigenvalues into the first equation in the original system of equations we obtain

$$(5-4)a_1 - 2a_2 = 0; \quad \mathbf{a}_1 = t_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{6}$$

and

$$(5-9)a_1 - 2a_2 = 0; \quad \mathbf{a}_2 = t_2\begin{bmatrix} 1 \\ -2 \end{bmatrix} \tag{7}$$

where t_1 and t_2 are arbitrary scalar multipliers, $t_1 \neq 0$, $t_2 \neq 0$. Had we substituted the eigenvalues into the second equation instead, the results would be the same. Computation of the product of the two eigenvectors yields

$$\mathbf{a}_1^T\mathbf{a}_2 = t_1 t_2 \begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 0 \tag{8}$$

The fact that the product is 0 means that the eigenvectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are orthogonal.

Exercises

1.1 Begin with the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 8 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 & -2 & 4 \\ 1 & 0 & 2 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 3 \\ 4 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

and perform the following matrix operations manually.
(a) $\mathbf{A} + \mathbf{B}$
(b) $\mathbf{A}\mathbf{B}^T$
(c) $\mathbf{B}^T\mathbf{A}$
(d) $\mathbf{A}\mathbf{C}$
(e) $\det \mathbf{C}$

1.2 Introduce the matrices

$\mathbf{A}$ with dimensions 4×3

$\mathbf{B}$ with dimensions 3×6

$\mathbf{C}$ with dimensions 1×8

$\mathbf{D}$ with dimensions 6×1

Which of the following operations are possible to perform? For the possible operations, give the dimensions of $\mathbf{E}$
(a) $\mathbf{E} = \mathbf{A}\mathbf{B}$
(b) $\mathbf{E} = \mathbf{B}\mathbf{D}$
(c) $\mathbf{E} = \mathbf{A}\mathbf{B}\mathbf{C}\mathbf{D}$
(d) $\mathbf{E} = \mathbf{A}\mathbf{B}\mathbf{D}\mathbf{C}$
(e) $\mathbf{E} = \mathbf{B}^T\mathbf{A}^T$

1.3 Solve the following system of equations manually. Check the solution.

$$\begin{bmatrix} 20 & 1 & -10 \\ -10 & 3 & 10 \\ 5 & 3 & 5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 9 \end{bmatrix}$$

1.4 Solve the following systems of equations manually and check the solutions.

(a)
$$\begin{bmatrix} 4 & -2 & -2 \\ -2 & 5 & -3 \\ -2 & -3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ a_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ 10 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 6 & -4 & -2 \\ -4 & 12 & -8 \\ -2 & -8 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ 16 \\ -6 \end{bmatrix}$$

(c) $$\begin{bmatrix} 4 & -4 & 0 & 0 & 0 \\ -4 & 7 & -2 & -1 & 0 \\ 0 & -2 & 5 & -3 & 0 \\ 0 & -1 & -3 & 7 & -3 \\ 0 & 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ a_2 \\ 0 \\ a_4 \\ 3 \end{bmatrix} = \begin{bmatrix} f_1 \\ 4 \\ f_3 \\ -1 \\ f_5 \end{bmatrix}$$

1.5 Begin with the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 & 3 \\ 6 & 4 & 1 & -2 \\ 0 & 3 & 4 & 1 \\ 1 & 2 & -4 & 6 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 3 & 4 & 1 & -2 \\ 6 & 8 & 1 & 0 \\ 2 & 2 & 3 & -2 \\ 1 & 4 & 0 & 4 \end{bmatrix};$$

$$\mathbf{C} = \begin{bmatrix} -4 \\ 2 \\ 3 \\ 1 \end{bmatrix}; \quad \mathbf{D} = \begin{bmatrix} 1 & 4 & -3 & 6 \end{bmatrix}$$

and perform the following matrix operations with CALFEM. For the sub-exercises with more than one matrix operation, compare and comment on the results.

(a) $\mathbf{A} + \mathbf{B}$ and $\mathbf{B} + \mathbf{A}$
(b) $\mathbf{AB}$ and $\mathbf{BA}$
(c) $(\mathbf{AB})^T$, $(\mathbf{BA})^T$ and $\mathbf{B}^T\mathbf{A}^T$
(d) $\mathbf{CD}$ and $\mathbf{DC}$
(e) $\mathbf{C}^T\mathbf{AC}$
(f) $\det \mathbf{A}$, $\mathbf{A}^{-1}$ and $\mathbf{AA}^{-1}$

1.6 Compute the determinant of the matrices in the following systems of equations with CALFEM. If possible, solve the systems of equations and check the solutions. If any of the systems is unsolvable, explain why.

(a) $$\begin{bmatrix} -4 & 3 & 0 & 1 \\ 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 2 \\ 2 & 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ -3 \\ 2 \end{bmatrix}$$

(b) $$\begin{bmatrix} 4 & -4 & 0 \\ -4 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(c) $$\begin{bmatrix} 8 & -3 & -5 \\ -3 & 5 & -2 \\ -5 & -2 & 7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -6 \end{bmatrix}$$

1.7 Consider the eigenvalue problem $(\mathbf{A} - \lambda\mathbf{I})\mathbf{a} = \mathbf{0}$, where

$$\mathbf{A} = \begin{bmatrix} 10 & -3 \\ -3 & 2 \end{bmatrix}$$

(a) Compute the eigenvalues.
(b) Compute the eigenvectors and check that they are orthogonal.

2

Systems of Connected Springs

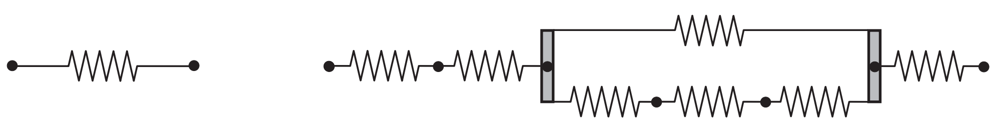

Figure 2.1 Elastic spring and a system of connected springs

A system of connected springs is a set of *discrete material points* connected by *springs* (Figure 2.1). Of the different building blocks of structural mechanics, the spring is the simplest one. The study of systems built up of springs only can therefore be an instructive way to describe and explain the models at the system level.

In structural mechanics, a system is basically composed of two components: *nodes with degrees of freedom* and *elements*. Here, we choose to study a system of connected springs that carries load only in one direction and we let this direction be the x-axis (Figure 2.2). A number of reference points or *nodes* are introduced. In each node, there can be an arbitrary number of *global degrees of freedom*. These degrees of freedom represent different possible movements for the ends of the elements connected to a node. Here, we choose to allow only one possible movement for each node, the displacement in a certain direction. The nodes and the degrees of freedom also form locations and directions where external forces (prescribed loads or arising support forces) can be applied and equilibrium equations can be set up.

Between two nodes, we can create a potential force path by inserting an elastic spring. The tendency of a spring to carry load depends on its *spring stiffness*. In a system with several different force paths, the stiffer ones carry the greatest load.

As was the case at the system level, the description of a single spring can be based on discrete nodes; here, they comprise the end points of the spring (Figure 2.3). To these *local nodes*, we can associate *local degrees of freedom*, which describe the possible movements of the nodes and also enable forces to act on the spring. Based on the degrees of freedom, defined for the element, a matrix that represents the stiffness properties of the spring is formed. This matrix can be placed between global degrees of freedom and constitutes then a force path in the global spring system.

Structural Mechanics: Modelling and Analysis of Frames and Trusses, First Edition.
Karl-Gunnar Olsson and Ola Dahlblom.

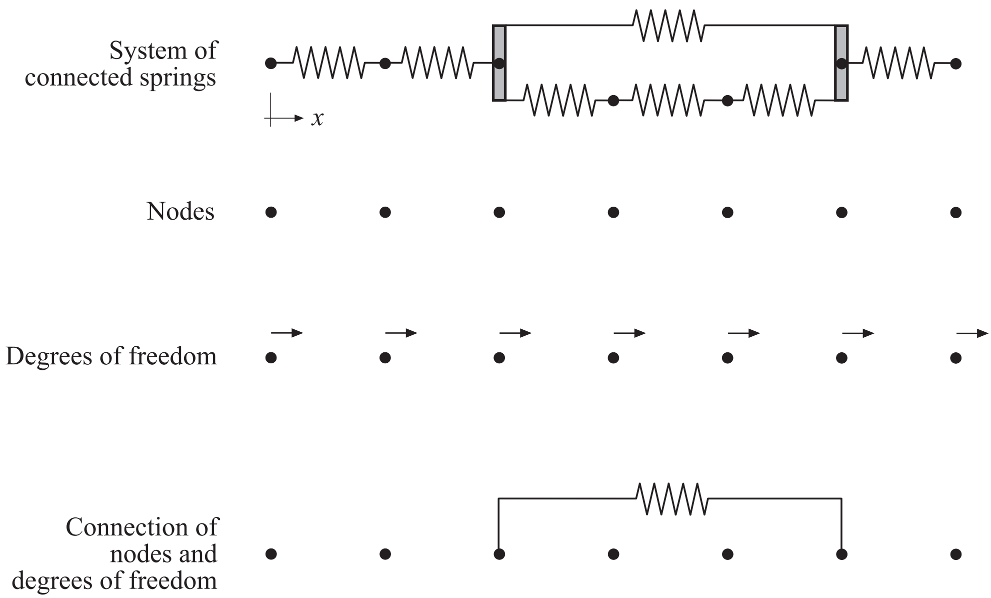

Figure 2.2 Nodes, degrees of freedom and connection of degrees of freedom

Figure 2.3 A spring element with two degrees of freedom

In structural mechanics, every system contains three basic quantities – force, stiffness and deformation – which can be considered at different scale levels. Figure 2.4 shows a map, which summarises the quantities and relations of a system of connected springs. The map has the following structure:

- a scale with three levels: the elastic spring, the systematically described spring element and the system of connected springs;
- three types of quantities: force measure, stiffness measure and displacement measure;
- for force measures: relations between force measures at different scale levels – equilibrium/ static equivalence;
- for displacement measures: relations between displacement measures at different scale levels – kinematics/compatibility;
- at each level: a constitutive relation between the force measure and corresponding displacement measure.

At the lowermost level, there is a relation between force and deformation for an *elastic spring*, $N = k\,\delta$. This relation is called the *constitutive relation* and is the basis for the derivation of corresponding relations at higher scale levels. The spring relation is further described in Section 2.1.

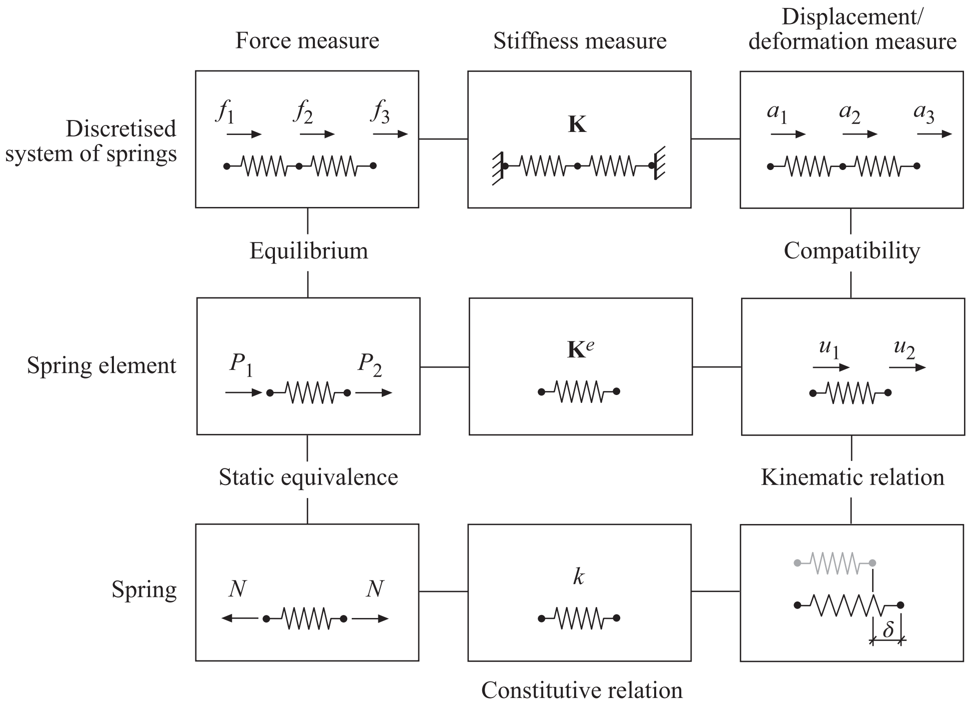

Figure 2.4 The quantities and relations of structural mechanics for springs and spring systems

By systematically introducing local degrees of freedom and expressing the deformation and the forces of the spring in connection to them, we can reformulate the constitutive relation of the spring to a corresponding constitutive relation for a *spring element*. This intermediate level, which is described in Section 2.2, is a preparatory step for the uppermost level of the scale, the model of a spring system.

The uppermost level deals with the systematic construction of computational models for global load-carrying structures. The methodology introduced here for a *system of connected springs* is general and is applied for all the systems considered in this book. The methodology consists of six steps, which are described in Section 2.3.

Each level in the map represents a constitutive relation between forces and deformations. Such a constitutive relation is always derived from a lower level to a higher one. We, in terms of six steps, introduce the general principle for such derivations.

- Start from the *constitutive relation* of the lower level (1).
- Define the deformation measure of the higher level, *kinematic quantities (2)*.
- Formulate a relation between the kinematic quantities of the lower and the higher level – *the kinematic relation* (3).
- Define the loading on the body/structure at the higher level, *force quantities* (4).
- Formulate a relation between the forces of the lower and the higher level – *equilibrium/static equivalence* (5).
- Determine a *constitutive relation* for the higher level using the three relations (6).

In Sections 2.1–2.3, the numbers of these steps recur in the text. Consistently throughout the textbook, each derivation from a lower to a higher level is concluded with a figure, which summarises Equations (1), (3) and (5), which lead to the constitutive relation of the higher level (6).

2.1 Spring Relations

The basic action of a spring is given by the relation

$$\boxed{N = k\ \delta} \tag{2.1}$$

which describes the resistance to deformation of a spring. The spring relation (Figure 2.5) consists of three types of quantities: the *force* N acting on the spring, the *stiffness* k of the spring and the *deformation* δ which arises. Equation (2.1) is the *constitutive relation* of the spring (1).

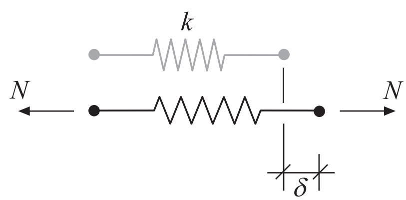

Figure 2.5 A spring with the stiffness k is loaded with the force N and thereby it is elongated by a distance δ

2.2 Spring Element

A discretised spring element (Figure 2.6) has two nodes, each with one displacement degree of freedom, u_1 and u_2. The displacements u_1 and u_2 are referred to as the *nodal displacements* of the element (2) and we choose here to define them as positive when they have the same direction as the x-axis. The forces acting at the nodes are denoted P_1 and P_2, and referred to as *element forces* (4). These are also defined to be positive in the direction of the x-axis.

We are now able to formulate a kinematic relation (3) by expressing the deformation δ of the spring as a function of the nodal displacements,

$$\boxed{\delta = u_2 - u_1} \tag{2.2}$$

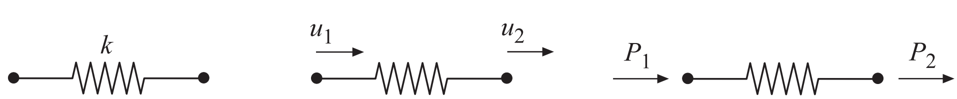

Figure 2.6 A discretised spring element