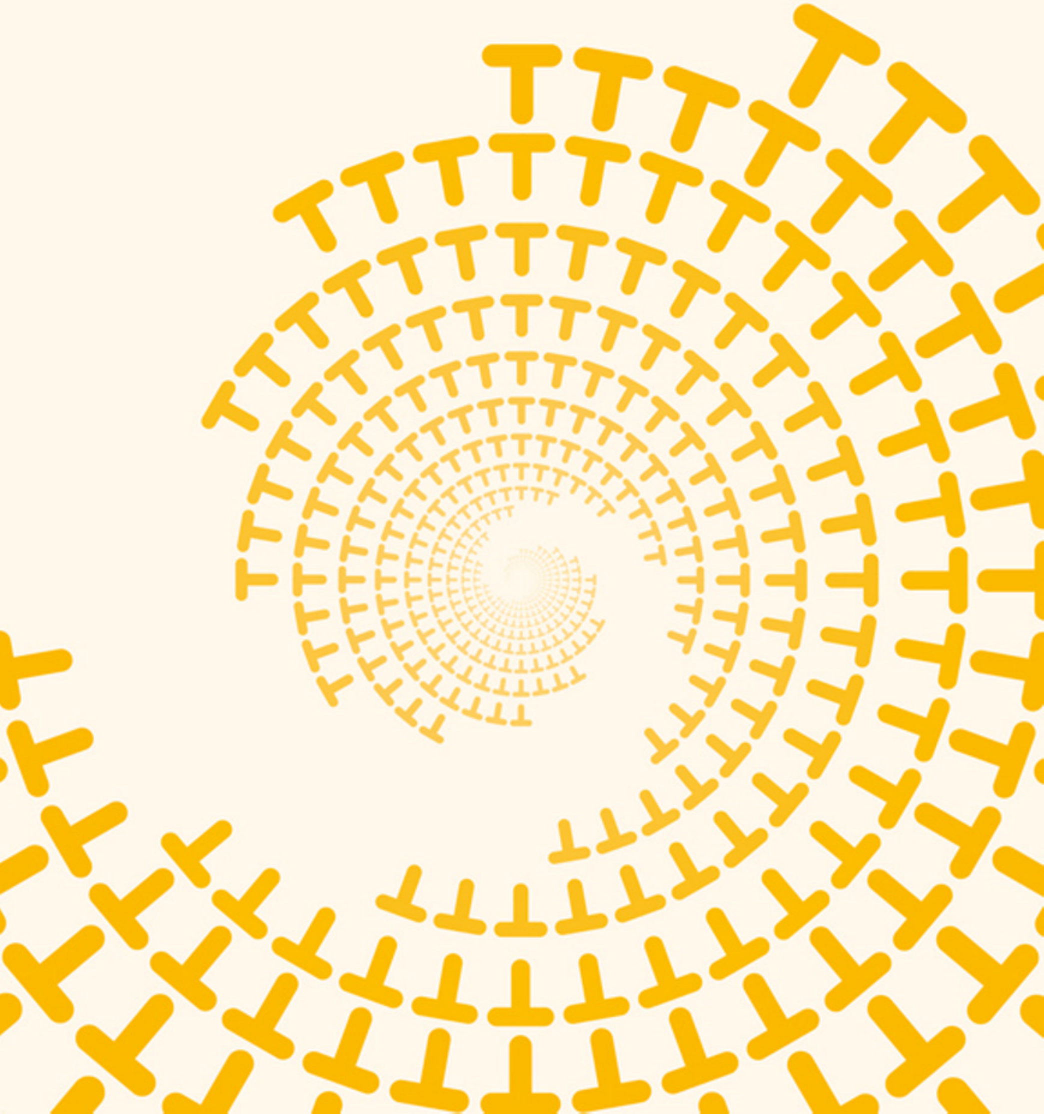
Paradoxes

Roy T. Cook



Paradoxes

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For Trust

and

for anyone whom I don’t discuss in this book.

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Introduction

The book you hold in your hands provides an overview of paradoxes, one accessible to readers who are not specialists in those fields that tend to pay a lot of (or at least some) attention to these puzzles. The goal is to present and discuss some paradoxes that have been, and in most cases continue to be, central concerns within philosophy and related disciplines such as mathematics, linguistics, and computer science. Thus, the issues discussed below will be of interest to students of and professionals in these disciplines. Paradoxes, however, are in one sense nothing more than extremely clever puzzles, and so it is hoped that the material covered in the chapters to follow will be of interest to a much wider audience than merely specialists in the areas just mentioned.

The observation that paradoxes are a species of puzzle should not lead the reader to conclude that they are not important. On the contrary, while paradoxes are in *one* sense merely extremely clever puzzles, in another sense they are among the most important puzzles ever devised. Paradoxes often demonstrate, or at least suggest, that our most basic intuitions and platitudes regarding some of our most basic concepts – including truth, collection, logic, knowledge, and belief – are faulty in some sense or another. As a result, extending our understanding of and (if we are lucky) providing solutions to these puzzles not only provides an entertaining diversion (and this book would never have been written did I not find paradoxes entertaining), but often leads to important new insights and entirely new approaches to these concepts (in some cases, entirely new disciplines devoted to the study of these concepts arise). For example, large parts of mathematics and mathematical logic owe their origin to ruminations on the Liar Paradox (Chapters 2 and 3) and the set-theoretic paradoxes (Chapter 4). Our understanding of how language works and our mathematical models of this understanding as developed within linguistics and the philosophy of language has benefited from thinking about the Liar Paradox (Chapters 2 and 3), and from thinking about vague predicates and the Sorites Paradox to which they seem to succumb (Chapter 5). And both psychology and the philosophical study of knowledge – that is, epistemology – owe great debts to the paradoxes involving knowledge and belief (Chapter 6).

Some philosophers (e.g. Sorensen 2005) have argued that the entire history of philosophy can be seen as a sequence of responses to various paradoxes (it is worth noting that Sorensen understands the term "paradox" to apply more widely than I do; see Chapter 1 for further discussion). Since most intellectual disciplines – arguably, all intellectual fields other than mathematics, law, and religion – were originally subdisciplines of philosophy (for example, Isaac Newton did not think of himself as a scientist, but as a natural philosopher), this would entail that the vast majority of intellectual inquiry of any sort can, in the end, be traced back to paradoxes. This is, of course, a bold and controversial claim, and I will not try to defend it here. I do find this view of this history of thought plausible, however, and the mere fact that such a position can be coherently argued for, whether right or wrong, is already enough to demonstrate the importance of paradoxes in the history of philosophy in particular and in intellectual progress more generally.

As we shall see in Chapter 1, a paradox is a particular type of argument, one that ends with an unacceptable conclusion of some sort. One of the main tasks of the chapters to follow is to convince the reader that paradoxes are not only interesting puzzles but also constitute real problems regarding our understanding of central and important concepts – problems that need to be addressed and solved. Given this way of viewing paradoxes – as symptoms of a deeper misunderstanding of the concepts involved – one natural way to approach paradoxes is in terms of the manner in which they are solved. As a result, we can understand solutions to paradoxes in terms of the various ways that the proposed solution 'defuses' the paradox. The solutions-oriented approach, outlined in Chapter 1, provides the framework for the remainder of the book.

We shall then spend the next five chapters examining a number of types of paradox, understanding various responses to and solutions to these puzzles in terms of the four general categories of solution outlined in Chapter 1. Importantly, not every solution to every paradox discussed above (much less those paradoxes not addressed here) falls precisely and unambiguously into one of the four categories of response outlined in Chapter 1. Nevertheless, the vast majority of such solutions do fall into one of our four categories (or into some hybrid combining two or more of these categories), and as a result the solutions-oriented approach provides a nice framework within which the majority of work on paradoxes can be situated.

It is worth noting that this book does not attempt to catalogue or taxonomize every paradox that has tormented philosophers, mathematicians, and the rest (Clark 2007 is one attempt at such a catalogue, covering a number of paradoxes not discussed here). Rather, the intent is to present a representative sample of paradoxes that are particularly important or particularly interesting. Of course, there is some risk that this selection is colored somewhat by my own interests. Even so, many paradoxes, of a number of different types, are covered in the sections to follow, and I feel confident that most readers will find many interesting conundrums in the resulting discussion.

Along similar lines, I do not attempt to catalogue every possible solution to each of the paradoxes discussed below. Even very superficial synopses of every solution to the Liar Paradox proposed during the twentieth century would require a book many times the size of the present one. What is attempted is to provide, for each of the types of paradox discussed in Chapters 2–6, examples of each of the four types of solution as outlined in Chapter 1. In short, the presentation of both paradoxes and their solutions is not meant to be exhaustive, but instead merely tries to present examples of most or all of the major types of paradox, and to provide representative examples of most or all of the main strategies or types of strategy for dealing with each type of paradox.

As already noted, the book is divided into six main chapters, plus the introduction you are now reading and a short concluding section. Chapter 1 presents an account of what paradoxes are and what approaches exist for dealing with them, and some well-known paradoxes (including paradoxes relating to theology, to infinity, and to infinite divisibility) are used to illustrate this taxonomy. Chapter 2 then examines one of the most well-known paradoxes – the *Liar Paradox* – a conundrum that shows that our intuitive understanding of truth (surely one of the most central and most important concepts for almost any inquiry) is somehow faulty. After this examination of the Liar Paradox itself, we examine some particularly troubling variants of the Liar Paradox in Chapter 3: The *Curry Paradox*, the *Yablo Paradox*, and the *Revenge Problem*. In Chapter 4 we shift our focus from truth and satisfaction to the concept of collection or set, examining the set-theoretical and infinitary paradoxes that plagued mathematics in the late nineteenth and early twentieth centuries. We then move on, in Chapter 5, to the paradoxes that arise due to vague predicates such as "is bald," "is red," or "is tall" – paradoxes that are known collectively as the *Sorites Paradox* or *Soritical Paradoxes*. Finally, in Chapter 6 we examine paradoxes involving epistemic notions such as knowledge and belief.

Two further things are worth noting about the organization and content of Chapters 2–6. First, the fact that two chapters are devoted to semantic paradoxes should not lead the reader to conclude that these paradoxes are twice as important or twice as difficult to solve as the puzzles discussed in the later chapters. On the contrary, most of the issues discussed in the second chapter on paradoxes involving truth also arise in some form or another with respect to paradoxes involving collections, vagueness, knowledge, and belief. These additional issues have received the most attention in the literature on semantic paradoxes, however, and introducing them in that context is therefore most natural.

Second, although I have segregated different types of paradox, involving different concepts, into distinct chapters, pains will be taken to point out connections between both the various paradoxes themselves and connections between various solutions to them. The reason for such care is a simple one: we need to determine whether these paradoxes are completely separate, unconnected maladies, or whether they are all simply different symptoms of some single, deeper disease. Whether one sees these paradoxes as completely distinct or as variations on a single theme will of course depend on whether one favors the same, or different, types of solution to different paradoxes. This theme will be examined a bit more explicitly in the concluding section of the book, where we will quickly look at the *Principle of Uniform Solution* (unfortunately acronymed PUS in the literature!). The Principle of Uniform Solution suggests that many if not all of the paradoxes discussed here should be solved in the same manner. In other words, applying the Principle of Uniform Solution to some class of paradoxes amounts to treating these paradoxes as stemming from a single underlying "mistake," and thus requires solving them in the same manner.

Finally, there are some general organizational issues that need to be noted. First, I have been sparing with bibliographic references in the text, only listing sources where a particular view or work is being directly quoted. For those readers who wish to track down the original sources of either the paradoxes or their solutions, however, I have included, at the end of each chapter, a list of useful further readings. Full citations for all works mentioned, either in the text or in the list of further readings, can be found in the references at the end of the volume.

Second, I have assumed that the reader is familiar with classical logic in at least an informal sense. Although I have provided schematic examples of particular classical inference rules and theorems in the text when relevant, for the most part it is assumed in what follows that the reader will be familiar with the general patterns of inference that govern logical operations such as "or," "and," "if…then…," and "if and only if" on the classical understanding. Since many of the solutions to paradoxes considered below involve rejecting one or another of the standard rules for classical logic (a rejection of the underlying logic is one of the four solution strategies introduced in Chapter 1), this background knowledge is absolutely critical for understanding many of the views discussed below. The reader who needs a refresher on classical logic will find no better source than Stewart Shapiro's article on the topic in the *Stanford Encyclopedia of Philosophy* (2009).

Third, I have included a list of seven exercises at the end of each main chapter (Chapters 1–6). As noted at the beginning of this introduction, paradoxes are immensely important, but they are also immensely fun. Some of this fun is lost when, instead of struggling with the puzzle and attempting to sort out the difficult issues on one's own, the puzzle and various proposed solutions to it are presented in essay form. To rectify this, the reader is encouraged to work through some or all of the exercises in order to 'get their hands dirty,' so to speak, working through variants of the conundrums discussed in the text.

Fourth, a word on notation: I have attempted, as far as is possible, to present the paradoxes below in the most accessible manner possible. It would have been wonderful if this meant that mathematical notation – in particular, the symbolic language of various formal logics – could have been avoided altogether. Unfortunately, it is impossible to present some of the material discussed below without the precision and efficiency provided by perspicuous notation. This is especially true of the discussion of Gödel's Incompleteness Theorems in Chapter 2 (and elsewhere), since these results are, strictly speaking, results about formal languages, and only apply to the informal natural languages modeled by formal constructions in an indirect way. In particular, in that discussion I have introduced the notation $\langle\Phi\rangle$ to denote a name of a linguistic expression Φ. $\langle\Phi\rangle$ refers either to the statement Φ enclosed in quotation marks, or to a numerical code for Φ, depending on the context. Additionally, I have used T(…), K(…), and B(…) as abbreviations for the truth, knowledge, and belief predicates, and $\Diamond$(…) for the possibility operator. I have throughout used the abbreviation:

not(Φ)

as shorthand for:

It is not the case that Φ.

Along similar lines, in Chapter 4 I have introduced some standard mathematical symbols for various constructions within set theory, including $\in$ for membership, $\subseteq$ for subsethood, and bracket notation:

$\{x : \Phi(x)\}$

for the set of objects that satisfy $\Phi(\ldots)$. These examples also illustrate my main convention regarding variables and schematic letters: Greek symbols will be used for these (such as when stating rules of inference or proving general results), and everyday Roman letters will be used when abbreviations are needed for particular statements.

Fifth, a note about the use of the term "infinity" is in order. As we shall see in Chapter 4, infinite collections can come in many different 'sizes.' In many of the examples and a number of the exercises (both before, during, and after Chapter 4), I will sometimes speak of an 'infinite set of statements' or 'an infinite sequence of gods' or 'an infinite collection of objects.' Unless noted otherwise, the reader should assume that this terminology refers to a sequence of objects that is ordered like the natural numbers:

$0, 1, 2, \ldots n - 1, n, n + 1, \ldots$

(Of course, the reader should also be open to the possibility that in some cases the puzzle might be solved by determining that there cannot be such an infinite sequence!) In short, unless the terminology of Chapter 4 is invoked to suggest otherwise, the reader should understand the term "infinite" in what follows to denote a countably infinite set, sequence, or list.

Finally, I have tried when possible to cite the original or canonical sources of paradoxes when they are discussed or appear in exercises. Unfortunately, some paradoxes have rather murky origins, and other paradoxes are so well known that their origins are rarely noted. As a result, I am relatively sure that there are paradoxes discussed below where I have failed to note the source. Of course, I have no idea which cases these are, else I would have corrected the problem (see the discussion of the Preface Paradox in Chapter 6 below!) Thus, I have no recourse except to offer pre-emptive apologies to anyone who deserves credit but fails to receive it in what follows.

1
The Care and Feeding of your New Paradoxes

In an episode of Matt Groening's *The Simpsons*, Homer Simpson asks Ned Flanders the following question:

> "Could Jesus microwave a burrito so hot that even he couldn't eat it?"

The puzzle, of course, if this: If we answer "no," then we have admitted that there is a task – microwaving a burrito so hot that it cannot be eaten – that Jesus could not even in principle perform, violating his supposed omnipotence. If we answer "yes," however, then we have again admitted that there is a task that Jesus cannot perform – namely, eating said burrito. Either way, we seem to be violating Jesus' omnipotence, and thus if Jesus really is omnipotent, then we seem stuck with a contradiction. This is a paradox (one known as the *paradox of omnipotence*, and more commonly formulated in terms of God creating a rock too heavy to lift).

A *paradox* (or *aporia*) is a type of argument. In particular, a paradox is an argument that:

(a) Begins with premises that seem uncontroversially true.
(b) Proceeds via reasoning that seems uncontroversially valid.

(c) Arrives at a conclusion that is a contradiction, is false, or is otherwise absurd, inappropriate, or unacceptable.

Conditions (a) and (b) are for the most part straightforward. Condition (c), however, deserves more comment.

(c) asserts that the problematic conclusion involved in a paradox must be a contradiction, false, or otherwise absurd or inappropriate. A contradiction is a statement that is not only false, but that *must* be false, where this guarantee holds in virtue of the logical, or grammatical, form of the statement. For example, any claim of the form:

Φ and not(Φ)

is a contradiction, since no statement can be both true and false (where Φ is a statement, "not(Φ)" abbreviates the grammatically more careful, but inconveniently unwieldy, statement "it is not the case that Φ"). Contradictions come in other flavors, however. In particular, I will in what follows sometimes make use of the underappreciated fact that (again, at least in standard classical treatments of logic) any statement of the form:

Φ if and only if not(Φ)

is also a contradiction.

Paradoxes need not result in outright contradictions, however. An argument will still be a paradox if the conclusion is false, but not a contradiction. For example, in Chapter 6 we will examine an argument (the Fitch Paradox) that purports to show that if all truths are knowable, then all truths are known. The claim that all truths are known is not a contradiction, since it does not describe a situation that is impossible (or, at least, doesn't seem to at first glance – see the discussion of blindspots in Chapter 6). The claim that all true statements are known is clearly false, however, which is enough to demonstrate that something must have gone wrong with the argument, and that the argument is therefore a paradox.

On the definition given above, paradoxes can also involve a conclusion that is neither a contradiction nor even a falsehood. There are paradoxes that consist of arguments based on apparently true premises that lead to conclusions that might be true, but which, in some very real sense, should not follow from the premises in question. For example, in Chapter 3 we will examine a paradox (the Curry Paradox) whose premises do nothing more than state conditions that truth ought to satisfy, and whose reasoning involves nothing more than basic inferences involving the expression "if…then…." Versions of this paradox can be constructed where the conclusion is, in fact, a true statement such as "Snow is white," or where the conclusion is a statement that could have been true such as "Santa Claus exists." The argument in this case is a paradox, not because "Snow is white" is a contradiction or a falsehood, but because we should not be able to demonstrate that snow is white based merely on considerations – that is, on premises – regarding philosophical concepts such as truth and on logical operations such as "if…then…."

Thus, paradoxes do not require conclusions that are contradictory, or even false. It is worth noting that there is something particularly disturbing about paradoxes whose conclusions take the form of a contradiction, however. In cases where the conclusion appears to be merely false, absurd or unacceptable, but not a contradiction, one strategy for dealing with the paradox is to decide that we were mistaken, and that the conclusion was not false, absurd, or unacceptable after all. Similarly, if the conclusion of a paradox is merely inappropriate (such as in the Curry Paradox "Snow is white" case above), one strategy is to re-evaluate what type of conclusions one should expect to follow from various sorts of premises (e.g. we might decide that questions about the color of objects really should follow from premises regarding truth and logic). The point is not that this sort of solution will always, or even often, be successful. On the contrary, in many if not most cases this sort of response is immensely implausible. The point, rather, is that in the case of paradoxes that involve a contradiction, this sort of response is unavailable as a matter of principle.

The problem with contradiction-involving paradoxes is deeper than the mere implausibility of accepting a contradiction as true. As we shall see, there are responses to paradoxes that make exactly this move. The problem is that this sort of response requires a solution to a further problem: the fact that a contradiction entails anything – at least, it does so in most accounts of logic and logical consequence. A theory that entails that every statement is true is called a *trivial theory*.

The argument that any theory that contains a contradiction is a trivial theory is simple. Assume that we have a contradiction of the form:

Φ and not Φ

We can now argue as follows: Given the truth of the offset statement above, it follows that Φ is true. So, for any statement Ψ whatsoever:

Φ or Ψ

is true. Since:

Not(Φ)

is also true, we can combine the previous two lines, via the rule of inference known as *disjunctive syllogism*:

$$\frac{\Omega \text{ or } \Theta \qquad \text{Not}(\Omega)}{\Theta}$$

to conclude that Ψ is true. A similar proof can be given if the contradiction is of the form:

Φ if and only if not(Φ)

As a result, anyone who accepts all of the inference rules used in the proof above – that is, *and-elimination* (or *adjunction*):

$$\frac{\Omega \text{ and } \Theta}{\Omega}$$

or-introduction (or *addition*):

$$\frac{\Omega}{\Omega \text{ or } \Theta}$$

and *disjunctive syllogism* – will also have to accept as valid the inference rule known as *explosion* or *ex falso quodlibet*:

$$\frac{\Omega \text{ and not}(\Omega)}{\Theta}$$

In short, in classical logic (and in many non-classical accounts of logic) anything follows from a contradiction.

Thus, anyone who wishes to accept the conclusion of a paradox involving a contradiction will need to reject one or more of the rules used above. Logics that reject the validity of *ex falso quodlibet* are known as *paraconsistent logics*, while logics that not only reject this classical rule but allow for the possibility that some contradictions are true are known as *dialethic logics*. As we shall see in Chapter 2, this latter sort of response to paradoxes – *dialetheism* – usually proceeds by denying disjunctive syllogism.

Before moving on, it is worth noting that other definitions of the notion of paradox have been offered. For example, Roy Sorensen defines paradoxes rather loosely as follows:

> I take paradoxes to be riddles. The oldest philosophical questions evolved from folklore and show vestiges of the verbal games that generated them. (2005: 3)

Sorensen understands a riddle to be a kind of question – one that typically has too many apparent answers. He fleshes this out a bit later in the same book:

> The riddle theory of paradox allows for the possibility of meaningless paradoxes. Riddles need only appear to be genuine questions; they can instead be meaningless utterances that look like questions. Pseudoquestions need only appear to have good answers and so need only appear to have an overabundance of good answers. (2005: 36)