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61

E. Bombieri (Ed.)

Geometric Measure Theory and Minimal Surfaces

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FONDAZIONE CIME ROBERTO CONTI

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Geometric Measure Theory and Minimal Surfaces

Lectures given at a Summer School of the
Centro Internazionale Matematico Estivo (C.I.M.E.),
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CENTRO INTERNAZIONALE MATEMATICO ESTIVO

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3° Ciclo - Varenna - dal 24 agosto al 2 settembre 1972

GEOMETRIC MEASURE THEORY AND MINIMAL SURFACES

Coordinatore: Prof. E. BOMBIERI

CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C.I.M.E.)

W. K. ALLARD

ON THE FIRST VARIATION OF AREA AND GENERALIZED MEAN CURVATURE

Corso tenuto a Varenna dal 24 agosto al 2 settembre 1972

W. K. Allard

Lecture One

Our object in these lectures is to describe the work of Almgren and the author on the first variation of the k dimensional area integrand in $\mathbb{R}^n$. We will work with a very general definition of k dimensional surface in $\mathbb{R}^n$ and will impose conditions on the first variations of the areas of these surfaces which will imply their rectifiability and differentiability.

We begin by giving a simple and very general definition of surface. Let $\mathbf{G}(n,k)$ be the Grassmann manifold of k dimensional linear subspaces of $\mathbb{R}^n$. Let $\mathbf{V}_k(\mathbb{R}^n)$ be the weakly topologized space of Radon measures on $\mathbb{R}^n \times \mathbf{G}(n,k)$. The elements of $\mathbf{V}_k(\mathbb{R}^n)$ are called k dimensional <u>varifolds</u> in $\mathbb{R}^n$. As we shall see, any k dimensional surface in $\mathbb{R}^n$ in the classical sense, with or without singularities, oriented or not, may be thought of as a k dimensional varifold in $\mathbb{R}^n$. Given $V \in \mathbf{V}_k(\mathbb{R}^n)$, we let $\|V\|(A) = V(A \times \mathbf{G}(n,k))$ for $A \subset \mathbb{R}^n$; evidently, $\|V\|$ is a Radon measure on $\mathbb{R}^n$.

Let $\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^n)$ be the algebra of linear endomorphisms of $\mathbb{R}^n$. We may identify $\mathbf{G}(n,k)$ with a compact nonsingular algebraic subvariety of $\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^n)$ by associating to any k dimensional linear subspace of $\mathbb{R}^n$, that is to any member of $\mathbf{G}(n,k)$, the endomorphism of $\mathbb{R}^n$ which orthogonally projects $\mathbb{R}^n$ onto this subspace. Thus, given $S \in \mathbf{G}(n,k)$, we will consider S as a linear subspace of $\mathbb{R}^n$ or a linear endomorphism of $\mathbb{R}^n$, whichever is convenient at the time. The space $\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^n)$ has a natural inner product given by the formula

$$A \cdot B = \text{trace } A^* \circ B \,, \qquad A, B \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n) \;;$$

here A^* is the adjoint of A .

Let $\mathcal{X}(\mathbb{R}^n)$ be the vector space of smooth compactly supported $\mathbb{R}^n$ valued functions on $\mathbb{R}^n$. In particular, if $g \in \mathcal{X}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ the total differential $Dg(x)$ is a member of $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$.

Let $V \in \mathbf{V}_k(\mathbb{R}^n)$. We define the <u>first variation distribution</u>

$$\delta V\colon \mathcal{X}(\mathbb{R}^n) \longrightarrow \mathbb{R}$$

by the formula

$$\delta V(g) = \int Dg(x) \cdot S \, dV(x,S) \,, \qquad g \in \mathcal{X}(\mathbb{R}^n) \,.$$

In the terminology of Laurent Schwartz, δV is a distribution on $\mathbb{R}^n$ of type $\mathbb{R}^n$. We now explain why we use the term "first variation". To do this we need to introduce the notions of <u>Jacobian</u> and <u>deformation</u>.

Suppose $F\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is smooth. We define the k dimensional Jacobian of F

$$J_k F\colon \mathbb{R}^n \times \mathbf{G}(n,k) \longrightarrow \{t\colon 0 \le t < \infty\}$$

by the formula

$$J_k F(x,S) = \frac{k \text{ area of } DF(x)[S \cap \{x\colon |x| < 1\}]}{k \text{ area of } S \cap \{x\colon |x| < 1\}} \,, \quad (x,S) \in \mathbb{R}^n \times \mathbf{G}(n,k) \,.$$

If one chooses an orthonormal basis $v_1, \ldots, v_k$ of S for which

$$\langle v_i, DF(x) \rangle \cdot \langle v_j, DF(x) \rangle = 0 \,, \qquad 1 \le i < j \le k \,,$$

one sees easily that

(1) $$J_kF(x,S) = \prod_{i=1}^{k} |\langle v_i, DF(x)\rangle| \ .$$

Moreover, it is not hard to define a homogeneous polynomial function P_k of degree $2k$ on $\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^n)$ such that

(2) $$J_kF(x,S)^2 = P_k(DF(x)\circ S), \qquad (x,S) \in \mathbb{R}^n \times \mathbf{G}(n,k) \ .$$

Now take $V \in \mathbf{V}_k(\mathbb{R}^n)$ and let

$$F_{\#}V(A) = \int_{\{(x,S):\ (F(x),DF(x)(S)) \in A\}} J_kF(x,S)dV(x,S) \ ,$$

$$A \subset \mathbb{R}^n \times \mathbf{G}(n,k) \ .$$

Then $F_{\#}V$ is a Borel regular measure on $\mathbb{R}^n \times \mathbf{G}(n,k)$. If, additionally, F is proper, it is clear that $F_{\#}V$ is a Radon measure on $\mathbb{R}^n \times \mathbf{G}(n,k)$; that is, $F_{\#}V \in \mathbf{V}_k(\mathbb{R}^n)$.

A triple (ε,h,K) is called a local deformation of $\mathbb{R}^n$ if $\varepsilon > 0$, $h: (-\varepsilon,\varepsilon) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is smooth, K is a compact subset of $\mathbb{R}^n$ and

$$h(t,x) = x \ \text{ if } \ t = 0 \quad \text{or} \quad x \notin K \ .$$

We set $h_t(x) = h(t,x)$ and $\dot{h}_t(x) = \frac{d}{du} h_{t+u}(x)\big|_{u=0}$ for $(t,x) \in (-\varepsilon,\varepsilon) \times \mathbb{R}^n$. Evidently, $\dot{h}_0 \in \mathcal{X}(\mathbb{R}^n)$ and h_t is a diffeomorphism of $\mathbb{R}^n$ for small t . Moreover, we have that

(3) $J_kh_t(x,S)$ is smooth in (t,x,S) for t near 0 ;

(4) $$\frac{d}{dt} J_kh_t(x,S)\Big|_{t=0} = D\dot{h}_0(x)\bullet S \ .$$

Indeed, (3) follows from (2) and (4) is verified by choosing an orthonormal basis $v_1, \ldots, v_k$ of S such that

$$\langle v_i, D\dot{h}_o(x)\rangle \cdot \langle v_j, D\dot{h}_o(x)\rangle = 0 , \qquad 1 \leq i < j \leq k ,$$

and then using (1) to calculate

$$\frac{d}{dt} J_k h_t(x,S)\Big|_{t=0} = \frac{d}{dt} \prod_{i=1}^{k} |v_i + t\langle v_i, D\dot{h}_o(x)\rangle|\Big|_{t=0}$$

$$= \sum_{i=1}^{k} \langle v_i, D\dot{h}_o(x)\rangle \bullet v_i$$

$$= D\dot{h}_o(x) \bullet S .$$

Using (3) and (4) we establish the following formula for $V \in \mathbf{V}_k(\mathbb{R}^n)$ and any local deformation (ε, h, K) of $\mathbb{R}^n$:

(5) $$\delta V(\dot{h}_o) = \frac{d}{dt} \|h_{t\#} V\|(K)\Big|_{t=0} ;$$

for this reason δV is called the first variation distribution of V .

We will now show how to associate a varifold, in a natural way, to any submanifold of $\mathbb{R}^n$ of locally finite area. We say M is a k <u>dimensional submanifold of class</u> p $(1 \leq p \leq \infty)$ <u>in</u> $\mathbb{R}^n$ if $M \subset \mathbb{R}^n$ and for every $a \in M$ there are class p functions φ: $\mathbb{R}^k \longrightarrow \mathbb{R}^n$ and ψ: $\mathbb{R}^n \longrightarrow \mathbb{R}^k$, and an open neighborhood W of a such that

$$\psi \circ \varphi(y) = y, \quad y \in \mathbb{R}^k \text{ and } W \cap M = W \cap \varphi(\mathbb{R}^k) .$$

Whenever $A \subset \mathbb{R}^n$ and $a \in \text{Closure } A$, we let $\text{Tan}(A,a)$ be the closed cone with vertex 0 in $\mathbb{R}^n$ consisting of those vectors v

in $\mathbb{R}^n$ such that either $v = 0$ or $v \neq 0$ and there are points $x_1, x_2, \ldots \in A \sim \{a\}$ such that $\lim_{i \to \infty} x_i = a$ and $\lim_{i \to \infty} |x_i - a|^{-1}(x_i - a) = |v|^{-1} v$. We let $\mathrm{Nor}(A,a) = \{w: v \cdot w \leq 0$ for all $v \in \mathrm{Tan}(A,a)\}$. Evidently, if M is as above,

$$\mathrm{Tan}(M,a) \in \mathbf{G}(n,k) \quad \text{and} \quad \mathrm{Nor}(M,a) \in \mathbf{G}(n,n-k)$$

for each $a \in M$. Let $\mathcal{H}^k$ be the k dimensional Hausdorff measure on $\mathbb{R}^n$; we set

$$|M|(A) = \mathcal{H}^k\{x: (x, \mathrm{Tan}(M,x)) \in A\}, \qquad A \subset \mathbb{R}^n \times \mathbf{G}(n,k),$$

and observe that $|M|$ is a Borel regular measure on $\mathbb{R}^n \times \mathbf{G}(n,k)$. Clearly, $|M| \in \mathbf{V}_k(\mathbb{R}^n)$ if and only if M intersects every bounded open subset of $\mathbb{R}^n$ in a set of finite k dimensional area. From the change of variables formula of advanced calculus we have that

(6) $\quad F_\# |M| = |F(M)|$ for any diffeomorphism F of $\mathbb{R}^n$.

This motivates the definition of $F_\#$. Suppose M is a k dimensional submanifold of class 1 in $\mathbb{R}^n$, $|M| \in \mathbf{V}_k(\mathbb{R}^n)$ and (ε, h, K) is a local deformation of $\mathbb{R}^n$; using (5) and (6) we see that

$$(7) \qquad \delta|M|(\dot{h}_0) = \frac{d}{dt} \mathcal{H}^k[h_t(M \cap K)]\Big|_{t=0}.$$

We now suppose that

(8) M is a smooth k dimensional submanifold of $\mathbb{R}^n$ with boundary B.

By this we mean that M is a k dimensional submanifold of class ∞

in $\mathbb{R}^n$, that $B = (\text{Closure } M) \sim M$, and that for each $b \in B$ there are smooth functions $\varphi: \mathbb{R}^k \longrightarrow \mathbb{R}^n$ and $\psi: \mathbb{R}^n \longrightarrow \mathbb{R}^k$, and an open neighborhood W of b such that

$$\psi \circ \varphi(y) = y, \quad y \in \mathbb{R}^k \quad \text{and} \quad W \cap M = W \cap \varphi(\mathbb{R}^k \cap \{y: y_k < 0\})$$

Clearly $|M| \in \mathbf{V}_k(\mathbb{R}^n)$. We will now calculate $\delta|M|$ in terms of the mean curvature vector of M and the exterior normal to M along B , which we now define.

Given $a \in M$, we define the bilinear function $B(a): \text{Tan}(M,a) \times \text{Tan}(M,a) \longrightarrow \text{Nor}(M,a)$, called the <u>second fundamental form</u> of M at a , by the requirement that

$$B(a)(v,w) \cdot u = -v \cdot \langle u, \langle w, \Psi \rangle \rangle , \quad u \in \text{Nor}(M,a) , \quad v, w \in \text{Tan}(M,a) ;$$

here $\Psi: \text{Tan}(M,a) \longrightarrow \text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ is the differential of $\text{Nor}(M,\cdot)$ at a , when $\text{Nor}(M,\cdot)$ is considered as a function on M with values in $\mathbf{G}(n,k) \subset \text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$. We have that

(9) $$v \cdot \langle w, Dg(a) \rangle = -B(a)(v,w) \cdot g(a) , \quad v, w \in \text{Tan}(M,a) ,$$
whenever $g \in \mathcal{X}(\mathbb{R}^n)$ and $g(x) \in \text{Nor}(M,x)$ for $x \in M$;

in fact, we may differentiate the equation $g(x) = \langle g(x), \text{Nor}(M,x) \rangle$, $x \in M$, in the direction w at a to obtain

$$\begin{aligned} v \cdot \langle w, Dg(a) \rangle &= v \cdot \langle \langle w, Dg(a) \rangle, \text{Nor}(M,a) \rangle + v \cdot \langle g(a), \langle w, \Psi \rangle \rangle \\ &= -B(a)(v,w) \cdot g(a) \end{aligned}$$

because $\langle \langle w, Dg(a) \rangle, \text{Nor}(M,a) \rangle \in \text{Nor}(M,a)$.

We define the <u>mean curvature vector</u> $H(a)$ of M at a by setting

$$H(a) = \frac{\operatorname{trace} B(a)}{k} \in \operatorname{Nor}(M,a) \ ;$$

from (9) we have immediately that

(10) $$Dg(a)\cdot\operatorname{Tan}(M,a) = -kg(a)\cdot H(a) \quad \text{whenever} \quad g \in \mathcal{X}(\mathbb{R}^n)$$
$$\text{and} \quad g(x) \in \operatorname{Nor}(M,x) \quad \text{for} \quad x \in M$$

Finally, given $b \in B$, we define $\nu(b) \in \mathbf{S}^{n-1} = \mathbf{R}^n \cap \{x: |x| = 1\}$ by the requirement

$$-\nu(b) \in \mathbf{S}^{n-1} \cap \operatorname{Tan}(M,b) \cap \operatorname{Nor}(B,b) \ .$$

We call $\nu(b)$ the <u>exterior normal</u> to M at b.

We have the following basic formula for $\delta|M|$:

(11) $$\delta|M|(g) =$$
$$= -k \int_M g(x)\cdot H(x)\, d\mathcal{H}^k x + \int_B g(b)\cdot\nu(b)\, d\mathcal{H}^{k-1} b, \quad g \in \mathcal{X}(\mathbb{R}^n).$$

We complete this lecture with the proof of this formula. In view of the existence of partitions of unity, it will suffice to verify that

(a) $$\delta|M|(g) = -k \int_M g(x)\cdot H(x)\, d\mathcal{H}^k x \quad \text{whenever} \quad g \in \mathcal{X}(\mathbf{R}^n)$$
$$\text{and} \quad g(x) \in \operatorname{Nor}(M,x) \quad \text{for} \quad x \in M$$

and that

(b) $$\delta|M|(g) = \int_B g(b)\cdot\nu(b)\, d\mathcal{H}^{k-1} b \quad \text{whenever} \quad g \in \mathcal{X}(\mathbf{R}^n) \ ,$$
$$g(x) \in \operatorname{Tan}(M,x) \quad \text{for} \quad x \in M \ ,$$

and for some $a \in M \cup B$ there are smooth functions $\varphi\colon \mathbb{R}^k \longrightarrow \mathbb{R}^n$ and $\psi\colon \mathbb{R}^n \longrightarrow \mathbb{R}^k$, and an open neighborhood W of a in $\mathbb{R}^n$, such that

$$\operatorname{spt} g \subset W, \qquad \psi\circ\varphi(y) = y \quad \text{for } y \in \mathbb{R}^k,$$

$$W \cap M = \begin{cases} W \cap \varphi(\mathbb{R}^k) & \text{if } a \in M \\ W \cap \varphi(\mathbb{R}^k \cap \{y\colon y_k < 0\}) & \text{if } a \in B. \end{cases}$$

Formula (a) follows immediately from (10). To prove (b), we let

$$\begin{aligned} \zeta_t(y) &= y + t \langle g\circ\varphi(y), (D\psi)\circ\varphi(y)\rangle, & (t,y) &\in \mathbb{R} \times \mathbb{R}^k; \\ h_t(y) &= x + \varphi\circ\zeta_t\circ\psi(x) - \varphi\circ\psi(x), & (t,x) &\in \mathbb{R} \times \mathbb{R}^n; \\ \alpha(y) &= J_k\varphi(y,\mathbb{R}^k), & y &\in \mathbb{R}^k. \end{aligned}$$

Because $h_t\circ\varphi = \varphi\circ\zeta_t$, the naturality of the Jacobian implies that

$$J_k(h_t\circ\varphi)(y,\mathbb{R}^k) = \alpha\circ\zeta_t(y) J_k\zeta_t(y,\mathbb{R}^k), \quad y \in \mathbb{R}^n;$$

with the help of (4) we compute

$$\frac{d}{dt} J_k(h_t\circ\varphi)(y,\mathbb{R}^k)\Big|_{t=0} = D(\alpha\dot{\zeta}_0)(y)\cdot\mathbb{R}^k, \qquad y \in \mathbb{R}^k.$$

Using (6) and (7) we see that

$$\delta|M|(g) = \begin{cases} \int_{\mathbb{R}^k} D(\alpha\dot{\zeta}_0)(y)\cdot\mathbb{R}^k \, d\mathcal{H}^k y, & \text{if } a \in M; \\ \int_{\{y\colon y_k < 0\}} D(\alpha\dot{\zeta}_0)(y)\cdot\mathbb{R}^k d\mathcal{H}^k y, & \text{if } a \in B. \end{cases}$$

If $a \in M$, $\delta|M|(g) = 0$ since $\alpha\dot{\zeta}_0 \in \mathcal{X}(\mathbb{R}^k)$, so (b) is verified in this case. Suppose now $a \in B$. For any $v \in \mathbb{R}^k$ and any $y \in \mathbb{R}^k$ with $y_k = 0$ we have that

$$\alpha(y)v\cdot e_k = J_{k-1}\varphi(y,\{y: y_k = 0\}) \langle v,D\varphi(y)\rangle \cdot \nu\circ\varphi(y) \;;$$

here e_k is the k'th standard basis vector in $\mathbb{R}^k$, and one verifies the equation easily by taking v to be a multiple of e_k and then orthogonal to e_k. Therefore,

$$\int_{\{y:\, y_k < 0\}} D(\alpha\dot{\zeta}_0)(y)\cdot \mathbb{R}^k \, d\mathcal{H}^k y =$$

$$= \int_{\{y:\, y_k = 0\}} \alpha(y)\dot{\zeta}_0(y)\cdot e_k \, d\mathcal{H}^{k-1} y$$

$$= \int_B g(b)\cdot\nu(b) \, d\mathcal{H}^{k-1} b$$

because $\langle\dot{\zeta}_0(y), D\varphi(y)\rangle = g\circ\varphi(y)$ for $y \in \mathbb{R}^k$.

Lecture Two

Suppose $V \in \mathbf{V}_k(\mathbb{R}^n)$. We define a Borel regular measure $\|\delta V\|$ on $\mathbb{R}^n$ as follows:

if U is an open subset of $\mathbb{R}^n$,

$\|\delta V\|(U) = \sup \{\delta V(g) : g \in \mathcal{X}(\mathbb{R}^n), |g| \leq 1 \text{ and } \operatorname{spt} g \subset U\}$;

if A is any subset of $\mathbb{R}^n$,

$\|\delta V\|(A) = \inf \{\|\delta V\|(U) : A \subset U \text{ and } U \text{ is open}\}$.

In other words, $\|\delta V\|$ is the total variation of the operator δV . Let us suppose that $\|\delta V\|$ is a Radon measure on $\mathbb{R}^n$; this means that for every bounded open subset U of $\mathbb{R}^n$ there is a constant C such that

$$\delta V(g) \leq C \sup\{|g(x)| : x \in \mathbb{R}^n\}$$

for every $g \in \mathcal{X}(\mathbb{R}^n)$ with $\operatorname{spt} g \subset U$.

It is then elementary that δV has a unique extension, also denoted δV , to the vector space of $\mathbb{R}^n$ valued bounded Baire functions on $\mathbb{R}^n$ with compact support, which satisfies the requirement that

$$\delta V(g) = \lim_{i \to \infty} \delta V(g_i)$$

whenever $g, g_1, g_2, \ldots$ are a uniformly bounded sequence of $\mathbb{R}^n$ valued Baire functions on $\mathbb{R}^n$ supported in some fixed compact set for which

$$\lim_{i \to \infty} g_i(x) = g(x) \quad \text{for all} \quad x \in \mathbb{R}^n .$$

As an example, if M,B,H,ν are as in (11) of Lecture One we have that

$$\|\delta|M|\|(K) = \int_{K \cap M} |H(x)|\,d\mathcal{H}^k x + \mathcal{H}^{k-1}(K \cap B)$$

for any compact subset K of $\mathbb{R}^n$.

The condition that $\|\delta V\|$ is a Radon measure, together with a certain "dimension axiom", implies that V is rectifiable; we say that a varifold V is rectifiable if there are continously differentiable k dimensional submanifolds $M_1, M_2, \ldots$ of $\mathbb{R}^n$ such that

$$V \leq \sum_{i=1}^{\infty} |M_i| .$$

(Note that we allow repetitions in the list $M_1, M_2, \ldots$.)

In order to formulate the "dimension axiom", we need to make a definition. Let $\alpha(k) = \mathcal{H}^k[\mathbb{R}^k \cap \{x: |x| < 1\}]$. Given $V \in \mathbf{V}_k(\mathbb{R}^n)$ and $a \in \mathbb{R}^n$, let

$$\Theta^k(\|V\|,a) = \lim_{r \downarrow 0} \frac{\|V\|\mathbb{B}(a,r)}{\alpha(k)r^k},$$

where $\mathbb{B}(a,r)$ is the closed ball centered at a of radius r. For example, if M,B are as in (8) of Lecture One,

$$\Theta^k(\||M|\|,a) = \begin{cases} 0 & \text{if } a \notin M \cup B ; \\ 1/2 & \text{if } a \in B ; \\ 1 & \text{if } a \in M . \end{cases}$$

If $V \in \mathbf{V}_k(\mathbb{R}^n)$ is rectifiable, $\Theta^k(\|V\|,\cdot)$ is a real valued $\mathcal{H}^k$ measurable function and

$$\|V\| = \mathcal{H}^k \llcorner \Theta^k(\|V\|,\cdot) \;;$$

clearly, $\Theta^k(\|V\|,x) > 0$ for $\|V\|$ almost all $x \in \mathbb{R}^n$; this is a basic fact in geometric measure theory. See [FE 2.10.19].

We now state precisely the

<u>Rectifiability Theorem.</u> Suppose $V \in \mathbf{V}_k(\mathbb{R}^n)$ and $\|\delta V\|$ is a Radon measure. Then

(a) $\Theta^k(\|V\|,x) \in \mathbb{R}$ for $\|V\|$ almost all $x \in \mathbb{R}^n$;

(b) if $\Theta^k(\|V\|,x) > 0$ for $\|V\|$ almost all $x \in \mathbb{R}^n$,

then V is rectifiable.

Our "dimension axiom" is that the density $\Theta^k(\|V\|,\cdot)$ be essentially positive; it says, roughly, that the dimension of the measure $\|V\|$ is at most k . We illustrate this condition by the following

<u>Example.</u> Take $k < n$ and choose a Radon measure μ on $\mathbf{G}(n,k)$. Let $V = \mathcal{H}^n \times \mu \in \mathbf{V}_k(\mathbb{R}^n)$. It is clear that V is not rectifiable. However, for any $g \in \mathcal{X}(\mathbb{R}^n)$, we have

$$\begin{aligned}\delta V(g) &= \int Dg(x)\cdot S \; dV(x,S)\\ &= \iint Dg(x)\cdot S \; d\mathcal{H}^n x \; d\mu S\\ &= \int \left(\int Dg(x)\, d\mathcal{H}^n x\right)\cdot S \; d\mu S\\ &= 0\end{aligned}$$

because $\int Dg(x)\, d\mathcal{H}^n x = 0$, g having compact support. Assuming $\mu(\mathbf{G}(n,k)) = 1$, we see that $\|V\| = \mathcal{H}^n$ so that

$$\Theta^k(\|V\|,x) = 0 \quad \text{for every} \quad x \in \mathbb{R}^n .$$

The positivity of the density is preserved by weak convergence in the sense of the following

Closure Theorem. Suppose $V_1, V_2, \ldots, V \in \mathbf{V}_k(\mathbb{R}^n)$, θ is a positive continuous function on $\mathbb{R}^n$

$$\lim_{i \to \infty} V_i = V \quad \text{in} \quad \mathbf{V}_k(\mathbb{R}^n)$$

$$\limsup_{i \to \infty} (\|V_i\| + \|\delta V_i\|)(K) < \infty$$

for every compact subset K of $\mathbb{R}^n$,

$\Theta^k(\|V_i\|, x) \geq \theta(x)$ for $\|V_i\|$ almost all $x \in \mathbb{R}^n$, $i = 1, 2, \ldots$.

Then

$$\Theta^k(\|V\|, x) \geq \theta(x) \quad \text{for } \|V\| \text{ almost all } x \in \mathbb{R}^n .$$

It is beyond the scope of these lectures to give a complete proof of these theorems. In the next lecture, however, we will derive all the geometric ingredients of their proofs.

W. K. Allard

Lecture Three

Suppose $V \in \mathbf{V}_k(\mathbb{R}^n)$ and $\|\delta V\|$ is a Radon measure on $\mathbb{R}^n$. For each $a \in \mathbb{R}^n$ and each $t \in \mathbb{R}$ we set

$$\zeta_{a,t}(x) = \begin{cases} x-a & \text{if } |x-a| \le t , \\ 0 & \text{if } t < |x-a| ; \end{cases}$$

$$\alpha_{a,V}(t) = \|V\|\{x: |x-a| \le t\} ; \qquad \beta_{a,V}(t) = \delta V(\zeta_{a,t}) ;$$

$$\gamma_{a,V}(t) = \int_{\{(x,S):\ 0 < |x-a| \le t\}} |x-a|^{-1} |S^{\perp}(x-a)|^2 \, dV(x,S) .$$

We have the basic relation about

Change of mass in concentric balls:

$$\text{(1)} \qquad \frac{s^{-k}\alpha_{a,V}(s)}{r^{-k}\alpha_{a,V}(r)} = \frac{\exp \int_r^s \frac{d\gamma_{a,V}(t)}{t\alpha_{a,V}(t)}}{\exp \int_r^s \frac{\beta_{a,V}(t)}{t\alpha_{a,V}(t)}\, dt}$$

whenever distance $(a, \operatorname{spt}\|V\|) < r < s < \infty$.

In proving (1) we suppose $a = 0$ and write α, β, γ for $\alpha_{o,V}$, $\beta_{o,V}$, $\gamma_{o,V}$, respectively. For each $\varepsilon > 0$ we choose a smooth function $f_\varepsilon : \mathbb{R}^n \longrightarrow \mathbb{R}$ in such a way that

$$\left.\begin{aligned} f_\varepsilon(x) &\longrightarrow |x| \\ |x| \operatorname{grad} f_\varepsilon(x) &\rightarrow x \end{aligned}\right\} \text{uniformly as } \varepsilon \downarrow 0 .$$

Let $\psi \in \mathcal{C}_o^\infty(\mathbb{R})$ and let $\varphi(t) = \int_t^\infty \psi(\tau)d\tau$, $t \in \mathbb{R}$. For each

$\varepsilon > 0$, let $g_\varepsilon(x) = \varphi(f_\varepsilon(x))x$, $x \in \mathbb{R}^n$. Note that $g_\varepsilon \in \mathcal{X}(\mathbb{R}^n)$ and that

$$\langle v, Dg_\varepsilon(x)\rangle = \varphi'(f_\varepsilon(x))v\cdot \text{grad } f_\varepsilon(x)x + \varphi(f_\varepsilon(x))v , \qquad v \in \mathbb{R}^n ,$$

so that

$$Dg_\varepsilon(x)\cdot S = \varphi'(f_\varepsilon(x)) \text{ grad } f_\varepsilon(x)\cdot x - $$
$$- \varphi'(f_\varepsilon(x)) \text{ grad } f_\varepsilon(x)\cdot S^\perp(x) + k\varphi(f_\varepsilon(x)) .$$

Integrating with respect to V and letting $\varepsilon \downarrow 0$, we have that

$$\int \varphi(t)d\beta(t) = \int \varphi'(t)t\, d\alpha(t) - $$
$$- \int \varphi'(t)d\gamma(t) + k \int \varphi(t)d\alpha(t) .$$

Integrating by parts in this last expression, we see that

$$\int \psi(t)\beta(t)dt = -\int \psi(t)t\, d\alpha(t) + $$
$$+ \int \psi(t)d\gamma(t) + k \int \psi(t)\alpha(t)dt$$

so that, in the sense of distribution theory,

$$\beta(t)dt = - t d\alpha(t) + d\gamma(t) + k\alpha(t)dt ;$$

$$t d\alpha(t) - k\alpha(t)dt = - \beta(t)dt + d\gamma(t) ;$$

$$\frac{d\alpha(t)}{\alpha(t)} - \frac{k}{t}\, dt = - \frac{\beta(t)dt}{t\alpha(t)} + \frac{d\gamma(t)}{t\alpha(t)} ;$$

$$d \log t^{-k}\alpha(t) = - \frac{\beta(t)dt}{t\alpha(t)} + \frac{d\gamma(t)}{t\alpha(t)} .$$

We integrate from r to s to obtain (1).

From (1) we draw two basic corollaries:

(2) $$r^{-k}\alpha_{a,V}(r) \leq s^{-k}\alpha_{a,V}(s) \exp \int_r^s \frac{\|\delta V\| \mathbb{B}(a,t)}{\|V\| \mathbb{B}(a,t)}\, dt$$

whenever $\text{distance}(a, \operatorname{spt}\|V\|) < r < s < \infty$;

(3) if $C \in \mathbf{V}_k(\mathbb{R}^n)$, $\delta C = 0$ and $r^{-k}\alpha_{o,C}(r)$ is constant as r varies, then $x \in S$ for C almost all (x,S) .

Both these statements follow almost immediately from (1). We first draw some consequences of (2). The first is that

(4) $$\Theta^k(\|V\|,a) \in \mathbb{R} \quad \text{whenever} \quad \limsup_{r \downarrow o} \frac{\|\delta V\| \mathbb{B}(a,r)}{\|V\| \mathbb{B}(a,r)} < \infty .$$

This is an immediate consequence of (2). Note that, as a consequence of the Besicovitch theory of symmetrical derivation ([FE 2.8, 2.9]), we have that

$$\lim_{r \downarrow o} \frac{\|\delta V\| \mathbb{B}(a,r)}{\|V\| \mathbb{B}(a,r)} \in \mathbb{R} \quad \text{for } \|V\| \text{ almost all } a \in \mathbb{R}^n ;$$

this is (a) of the Rectifiability Theorem of Lecture Two.

We have the following uppersemicontinuity property of the density: If

$$\lim_{i \to \infty} V_i = V \text{ in } \mathbf{V}_k(\mathbb{R}^n), \qquad \lim_{i \to \infty} a_i = a \text{ in } \mathbb{R}^n ,$$

and for some $\varepsilon > 0$

$$\varepsilon \|\delta V_i\| \mathbb{B}(a_i,r) \leq \|V_i\| \mathbb{B}(a_i,r) , \qquad 0 < r \leq \varepsilon ,$$

then

(5) $$\limsup_{i \to \infty} \Theta^k(\|V_i\|,a) \leq \Theta^k(\|V\|,a) .$$

In fact, whenever $0 < r \leq \varepsilon$,

$$r^{-k}\alpha_{a,V}(r) \geq \limsup_{i \to \infty} r^{-k}\alpha_{a,V_i}(r)$$

and

$$\begin{aligned} r^{-k}\alpha_{a,V_i}(r) &\geq \\ &\geq (r-|a-a_i|)^{-k}\, \alpha_{a_i V_i}(r-|a-a_i|)\ (1-|a_i-a|/r)^k \\ &\geq \alpha(k)\Theta^k(\|V_i\|,a_i)\ \exp(-r/\varepsilon)\ (1-|a_i-a|/r)^k \ ; \end{aligned}$$

we let $i \to \infty$ and then let $r \downarrow 0$ to obtain (5).

A very important consequence of (5) is the

Isoperimetric Inequality. Suppose $V \in \mathbf{V}_k(\mathbb{R}^n)$, $\|V\|(\mathbb{R}^n) < \infty$ and $\Theta^k(\|V\|,x) \geq 1$ for $\|V\|$ almost all $x \in \mathbb{R}^n$. Then

(6) $$\|V\|(\mathbb{R}^n)^{(k-1)/k} \leq C\|\delta V\|(\mathbb{R}^n) \ .$$

Here C is a constant depending only on n .

The proof is as follows. Suppose $1 < \lambda < \infty$ and $s = (\lambda\|V\|(\mathbb{R}^n)/\alpha(k))^{1/k}$. If $a \in \mathbb{R}^n$ is such that $\Theta^k(\|V\|,a) \geq 1$ we have from (2) that

$$\exp \int_0^s \frac{\|\delta V\| \mathbb{B}(a,t)}{\|V\| \mathbb{B}(a,t)}\, dt \geq \frac{\Theta^k(\|V\|,a)\ \alpha(k)s^k}{\alpha_{a,V}(s)}$$

$$\geq \lambda$$

so that for some $t(a)$ with $0 < t(a) < s$

$$\frac{\|\delta V\| \mathbb{B}(a,t(a))}{\|V\| \mathbb{B}(a,t(a))} \geq \frac{\log \lambda}{s} \ .$$

The inequality (6) now follows from the covering lemma of Besicovitch in the form given by [FE 2.8.14].

An immediate corollary of (6) is that

(7) $\mathcal{H}^k(M)^{(k-1)/k} \leq C[\int_M |H(x)| d\mathcal{H}^k x + \mathcal{H}^{k-1}(B)]$

whenever M, B, H are as in (8) of Lecture One

<u>and</u> $\mathcal{H}^k(M) < \infty$.

Using the inequality (6), one can prove a Sobolov type inequality for varifolds, and consequently for manifolds; we omit the details.

Let us now consider the assertion (3). We assert that if C is as in (3), the measure $\|C\|$ is homogeneous of degree k, that is

(8) $$\int \varphi(x) d\|C\|x = r^{-k} \int \varphi(rx) d\|C\|x$$

whenever $0 < r < \infty$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$.

To verify this, suppose $f: \mathbb{R}^n \sim \{0\} \longrightarrow \{t: 0 \leq t < \infty\}$ is smooth and homogeneous of degree 0 so that

$$\text{grad } f(x) \cdot S(x) = 0 \text{ for } C \text{ almost all } (x,S) .$$

Let $V_f \in \mathbf{V}_k(\mathbb{R}^n)$ be characterized by the condition that

$V_f(A) = \int_A f(x) dV(x,S)$ for every Borel subset A of $\mathbb{R}^n \times \mathbf{G}(n,k)$.

One readily verifies that $\delta C_f(g_\varepsilon) = 0$ for g_ε as in the proof of (1) and argues as in the proof of (1) that $t^{-k}\alpha_{o,C_f}(t)$ is constant as a function of t since $\beta_{o,C_f}(t) = \gamma_{o,C_f}(t) = 0$, $0 < t < \infty$. The relation (8) is now a technical consequence of these observations.