Giorgio Ferrarese (Ed.)

CIME Summer Schools

Wave Propagation

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Bressanone, Italy 1980

Springer

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Lectures given at a Summer School of the
Centro Internazionale Matematico Estivo (C.I.M.E.),
held in Bressanone (Bolzano), Italy,
June 8-17, 1980

Springer

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ISBN 978-3-642-11064-1 e-ISBN: 978-3-642-11066-5
DOI:10.1007/978-3-642-11066-5
Springer Heidelberg Dordrecht London New York

Reprint of the 1st Ed. C.I.M.E., Ed. Liguori, Napoli & Birkhäuser 1982
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CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C.I.M.E.)

LECTURES ON NONLINEAR WAVE PROPAGATION

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CIME Session on Wave Propagation

Bressanone, June 1980

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Lecture 1. Fundamental Ideas Concerning Wave Equations

1. General Ideas

The physical concept of a wave is a very general one. It includes the cases of a clearly identifiable disturbance, that may either be localised or non-localised, and which propagates in space with increasing time, a time-dependent disturbance throughout space that may or may not be repetitive in nature and which frequently has no persistent geometrical feature that can be said to propagate, and even periodic behaviour in space that is independent of the time. The most important single feature that characterises a wave when time is involved, and which separates wave-like behaviour from the mere dependence of a solution on time, is that some attribute of it can be shown to propagate in space at a finite speed.

In time dependent situations, the partial differential equations most closely associated with wave propagation are of hyperbolic type, and they may be either linear or nonlinear. However, when parabolic equations are considered which have nonlinear terms, then they also can often be regarded as describing wave propagation in the above-mentioned general sense. Their role in the study of nonlinear wave propagation is becoming increasingly important, and knowledge of the properties of their solutions, both qualitative and quantitative, is of considerable value when applications to physical problems are to be made. These equations frequently arise as a result of the determination of the asymptotic behaviour of a complicated system.

Nonlinearity in waves manifests itself in a variety of ways, and in the case of waves governed by hyperbolic equations, perhaps the most striking is the evolution of discontinuous solutions from arbitrarily well behaved initial data. In the case of parabolic equations the effect of nonlinearity is tempered by the effects of dissipation and dispersion that might also be present. Roughly speaking, when the dispersion effect is weak, long wave behaviour is possible, whereas when it is strong a highly oscillatory behaviour occurs, though the envelope of the oscillations then exhibits some of the characteristics of long waves.

Waves governed by a linear wave equation arise in many familiar physical situations, like electromagnetic theory, vibrations in linear elastic solids, acoustics and in irrotational inviscid liquids. However, these linear equations often arise as a consequence of an approximation involving small amplitude waves, so that when this assumption is violated the equations governing the motion become nonlinear.

Not only does this convert the problem to one involving nonlinear partial differential equations, but it also usually leads to the study of a system of first order equations, rather than to a nonlinear form of the familiar second order wave equation. This happens because the wave equation usually arises as the result of the elimination of certain dependent variables from first order equations (like $\underline{E}$ or $\underline{H}$ in electromagnetic theory), and this is often impossible when nonlinearity arises. Our concern hereafter will thus be mainly with quasilinear first order systems of equations - that is to say with systems that are linear in their first order derivatives, and for the most part we will confine attention to one space dimension and time.

2. The Linear Wave Equation

Because of the importance of the linear wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u \,, \tag{1}$$

let us begin by reviewing some of the basic ideas that are involved, though in the more general context of the variable coefficient equation

$$\sum_{i,j=0}^{3} a_{ij}\, u_{x^i x^j} + \sum_{i=0}^{3} b_i u_{x^i} + cu = f \tag{2}$$

with a_{ij}, b_i, c, f functions of the four dimensional vector $\underline{s} = (x^0, x^1, x^2, x^3)$. Not all linear second-order equations of this form describe wave motion, and on account of this it is necessary to produce a method of classification which readily allows the identification of wave type equations from amongst the other types that are possible (i.e. elliptic and parabolic).

The form of classification to be adopted utilizes the coefficients of the highest-order derivatives and has an algebraic basis but, as will be seen in a subsequent section, this classification in fact effectively distinguishes between equations of wave type and those of other types. Let us start by attempting some simplification of the form of equation (2) by changing the independent variables through the linear transformation

$$\xi^i = \sum_{i=0}^{3} \alpha_{ij} x^j , \quad i = 0,1,2,3 \tag{3}$$

where the α_{ij} are constants.

A transformation of this form gives an affine mapping of the (x^0, x^1, x^2, x^3)-space which is one-one provided $\det|\alpha_{ij}| \neq 0$. Employing the chain rule for differentiation we find that equation (2) may be re-written

$$\sum_{i,j,k,\ell=0}^{3} a_{ij}\alpha_{ki}\alpha_{\ell j} u_{\xi^k \xi^\ell} + \sum_{i,k=0}^{3} b_i \alpha_{ki} u_{\xi^k} + f = 0 . \tag{4}$$

Hence the coefficients a_{ij} of the derivatives $u_{x^i x^j}$, which are functions of position, transform to the new coefficients

$$\tilde{a}_{k\ell} = \sum_{i,j=0}^{3} a_{ij}\alpha_{ki}\alpha_{\ell j}$$

of the derivatives $u_{\xi^k \xi^\ell}$, which are also functions of position. If, now, we confine attention to the set of coefficients a_{ij} appropriate to some specific point $\underline{s} = \underline{s}_0$ in (x^0, x^1, x^2, x^3)-space, we see that this is exactly the transformation rule which would apply to the coefficients a_{ij} of the quadratic form

$$\sum_{i,j=0}^{3} a_{ij}\eta_i\eta_j , \tag{5}$$

when the η_i are transformed to θ_k by the variable change

$$\eta_i = \sum_{k=0}^{3} \alpha_{ki}\theta_k .$$

Now it is a standard algebraic result that by a suitable transformation a quadratic form with constant coefficients may always be reduced to a sum of squares, though not all of the squared terms need be of the same sign. Furthermore, Sylvester's law of inertia asserts that however this reduction is accomplished, the number of positive terms m and the number of negative terms n will always be the same. To apply these results to the differential equation (2) itself with the variable coefficients a_{ij}, let us again confine attention to a fixed point $\underline{s} = \underline{s}_0$ in (x^0, x^1, x^2, x^3)-space and attribute to the a_{ij} the specific values $a_{ij} = a_{ij}(\underline{s}_0)$.

This then implies that some choice of the numbers $\alpha_{ij} = \tilde{\alpha}_{ij}$ exists for which

$$\sum_{i,j=0}^{3} a_{ij}\eta_i\eta_j = \sum_{i=0}^{m-1} \theta_i^2 - \sum_{i=m}^{m+n-1} \theta_i^2 ,$$

where $m + n \leq 4$. The number pair (m,n) is called the signature of the quadratic form (5) and, being an algebraic invariant, is used to classify the quadratic form. We shall use it to classify the variable coefficient partial differential equation (2) at each point $\underline{s} = \underline{s}_0$.

The effect on equation (2) of using these numbers $\tilde{\alpha}_{ij}$ in the transformation (3) is to yield at $\underline{s} = \underline{s}_0$ a differential equation of the form

$$\sum_{i=0}^{m-1} u_{\xi^i\xi^i} - \sum_{i=m}^{m+n-1} u_{\xi^i\xi^i} + \sum_{i=0}^{3} \tilde{b}_i u_{\xi^i} + f = 0 \qquad (6)$$

Equation (6) or, equivalently, (2) is called hyperbolic at $\underline{s} = \underline{s}_0$ in the ξ^0-direction when the signature is $(1,3)$, elliptic when the signature is $(4,0)$ and parabolic when $m + n < 4$. If an equation is hyperbolic in the ξ^0-direction at each point of a region Ω, then it is said to be hyperbolic in the ξ^0-direction throughout Ω.

Obviously, if an equation has constant coefficients, then one suitable transformation (3) will reduce it to the form of equation (6) throughout all space. For example, aside from the trivial transformation to remove the constant factor $1/c^2$, the wave equation (1) is already seen to have the signature $(1,3)$. Thus if a transformation is made at one point of space to convert the factor $1/c^2$ to unity, then it does so for all points in the space.

The usual effect of variable coefficients and first-order terms in hyperbolic equations of the form (2) is to introduce distortion as the wave profile propagates. This produces various complications, not the least of which is the fact that the wave velocity becomes ambiguous and requires careful definition. Only when there is a clearly identifiable feature of the wave which is preserved throughout propagation is it possible to define the propagation speed of this feature unambiguously. Such is the case with a wave front separating, say, a disturbed and an undisturbed region and across which a derivative of the solution is discontinuous.

3. The Cauchy Problem - Characteristic Surfaces

Fundamental to the study of hyperbolic equations is the Cauchy problem, and the associated notion of a characteristic surface. In brief, when working with four independent variables the Cauchy problem amounts to the determination of a unique solution to an initial value problem in which a hypersurface F is given, and on it the function u is specified together with the derivative of u along some vector directed out of F. Such a directional derivative is called an exterior derivative of u with respect to F, in order to distinguish it from a directional derivative in F which is known as an interior derivative. In the Cauchy problem it must be emphasized that the function u and its exterior derivative over the initial hypersurface F are independent, and can be specified arbitrarily.

A hypersurface F for which the Cauchy problem is not meaningful because u and its exterior derivative cannot be specified independently is called a characteristic hypersurface. Let us now see how characteristic hypersurfaces may be determined.

It is convenient to utilize curvi-linear coordinates ξ^0, ξ^1, ξ^2, ξ^3 and to let the hypersurface F on which the initial data is to be specified have the equation $\xi^0 = 0$. In terms of the new variables, a derivative with respect to ξ^0 is a directional derivative normal to F so that it is an exterior derivative, whilst derivatives with respect to ξ^1, ξ^2, ξ^3 are interior derivatives.

We now utilize this by rewriting equation (2) in a form in which the derivative $u_{\xi^0\xi^0}$ is separated from the other second-order derivatives

$$\left(\sum_{i,j,=0}^{3} a_{ij} \frac{\partial \xi^0}{\partial x^i} \frac{\partial \xi^0}{\partial x^j}\right) u_{\xi^0 \xi^0} + \sum_{i,j,k,\ell=0}^{3}{}' a_{ij} \frac{\partial \xi^k}{\partial x^i} \frac{\partial \xi^\ell}{\partial x^j} u_{\xi^k \xi^\ell}$$

$$+ \sum_{i,k=0}^{3} b_i \frac{\partial \xi^k}{\partial x^i} u_{\xi^k} + cu = f . \tag{7}$$

Here $\sum'$ signifies that the terms corresponding to $k = \ell = 0$ are omitted from the summation.

Now if we specify u and u_{ξ^0} independently on F, as is required in the Cauchy problem, the substitution of their functional forms into equation (7) enables the determination of $u_{\xi^0 \xi^0}$, provided only that the coefficient of this derivative does not vanish. Thereafter, the solution may be obtained in the form of a Taylor series by determining the coefficients of the series by successive differentiation of the initial data and of equation (7) itself. This is, of course, the idea underlying the classical Cauchy-Kowalewski theorem. It is, however, very restrictive as an existence theorem since it demands that all functions involved are C^∞.

In the event that the coefficient

$$\sum_{i,j=0}^{3} a_{ij} \frac{\partial \xi^0}{\partial x^i} \frac{\partial \xi^0}{\partial x^j} \tag{8}$$

of $u_{\xi^0 \xi^0}$ vanishes, neither this nor higher-order derivatives of u with respect to ξ^0 can be found. Furthermore, the derivative $u_{\xi^0 \xi^0}$ may then be specified arbitrarily on F, and even differently on opposite sides of F. This is not remarkable, because when the coefficient of $u_{\xi^0 \xi^0}$ vanishes, u and u_{ξ^0} cannot be specified independently over F. This follows because they must satisfy the equation which results when the first term is deleted from equation (7), and so we then have insufficient initial data.

As already mentioned, the hypersurface F with the equation $\xi^0 = 0$ for which the coefficient (8) vanishes is called a <u>characteristic hypersurface</u> of the differential equation (2). To examine such hypersurfaces further, we begin by setting $p_i = \partial \xi^0 / \partial x^i$ and writing

$$H \equiv \sum_{i,j=0}^{3} a_{ij} p_i p_j . \tag{9}$$

Then the quadratic form H is the coefficient of the derivative $u_{\xi^0\xi^0}$ in equation (7), and the characteristic hypersurface F will be given by the condition H = 0.

To interpret the condition H = 0, we first recall that if Ψ is a differentiable scalar function, then gradΨ is a vector normal to the surface Ψ = const. Consequently, by analogy, $p_i = \partial\xi^0/\partial x^i$ is the ith component of the four-dimensional gradient of ξ^0 and so is the ith component of a four-dimensional vector $\underline{p}$ normal to the hypersurface F. Hence the equation H = 0 is a condition on the orientation of the normal vector $\underline{p}$ to F, and as the a_{ij} are usually functions of position, it follows that this condition will differ from point to point.

The quadratic form (9) is, of course, just the same quadratic form we encountered in (5), so that its signature will depend on the type of the equation (7) or, equivalently, (2). If the equation is hyperbolic at $\underline{s} = \underline{s}_0$ the signature will be (1,3), and it follows that at the point the condition H = 0 determining the characteristic hypersurface can be reduced to

$$p_0^2 = p_1^2 + p_2^2 + p_3^2 \; . \tag{10}$$

It is obvious that no real characteristic hypersurface exists for elliptic equations, since their signature is (4,0) and the components of the vector $\underline{p}$ need to be complex if they are to satisfy the condition

$$H = p_0^2 + p_1^2 + p_2^2 + p_3^2 \;=\; 0 \; .$$

To proceed with the hyperbolic case we now simplify matters by setting $x^0 = t$ and writing

$$\xi^0 \;=\; t - \phi(x^1,\, x^2,\, x^3) \tag{11}$$

so that $p_0 = 1$ and $p_i = -\phi_{x^i}$ for i = 1,2,3. Then the quadratic form (10) becomes

$$\phi^2_{x^1} + \phi^2_{x^2} + \phi^2_{x^3} \;=\; 1 \tag{12}$$

which is a differential equation for the function ϕ locally at $\underline{s} = \underline{s}_0$. This is, of course, the familiar Eikonal equation from mathematical physics. At any time $t = t_0$ a real three-dimensional surface S is defined by

$$\phi(x^1, x^2, x^3) = t_0 \tag{13}$$

and this is called a characteristic surface.

If equation (7) is a constant coefficient equation it can be reduced to the form of equation (6) with $m = 1$, $n = 3$ throughout all space, so that equation (12) then describes the characteristic surface ϕ = const for all points in space.

In summary, we have established that real characteristic surfaces occur in connection with hyperbolic equations, and that across such surfaces a discontinuity may occur in the second normal derivative of the solution. This discontinuity in a derivative of a solution is usually identifiable with an interesting physical attribute of the solution, since it represents a wavefront bounding two regions.

The discontinuity surface, or wavefront, advances with time, as is shown by the following simple argument.

Taking the total differential of $\xi^0 = 0$ and using equation (11) we find

$$dt - dx^1\phi_{x^1} - dx^2\phi_{x^2} - dx^3\phi_{x^3} = 0$$

or, equivalently

$$dt = d\underline{r}.\text{grad}\phi ,$$

where $d\underline{r}$ is the vector differential with components (dx^1, dx^2, dx^3).

Hence

$$\frac{1}{|\text{grad}\phi|} = \underline{v}.\underline{n} \tag{14}$$

where

$$\underline{v} = \frac{d\underline{r}}{dt} , \quad \underline{n} = \frac{\text{grad}\phi}{|\text{grad}\phi|} .$$

The vector $\underline{n}$ is the unit normal to the surface ϕ = const, and as $d\underline{r}$ represents the displacement of a position vector with time, $\underline{v} = d\underline{r}/dt$ is the velocity of displacement of a specific point on the surface as the characteristic surface moves from its position at time t to its position at t + dt. The scalar $\underline{v}.\underline{n}$ is the normal velocity of propagation of the characteristic surface or wavefront and, in general, is a function of position.

By re-writing equation (7) and differencing it across the characteristic surface, we shall see that there may also be a discontinuity in the first normal derivative of the solution and this, like the discontinuity in the second-order derivative, is propagated with the characteristic surface. The equation governing the development of the discontinuities in first- and second-order derivatives is an ordinary differential equation defined along a curve in space and is called the transport equation.

4. Domain of Dependence - Energy Integral

The dependence of a wave solution on initial data is most easily illustrated in terms of the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 , \qquad (15)$$

with the initial conditions

$$u(x,0) = h(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = k(x) . \qquad (16)$$

The explicit d'Alembert solution

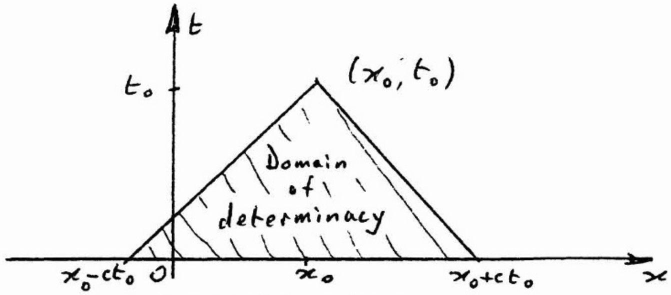
$$u(x,t) = \frac{h(x-ct)+h(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} k(s)\, ds \qquad (17)$$

shows how the solution at (x_0,t_0) depends only on data in the interval

$$x_0 - ct_0 \le x \le x_0 + ct_0 .$$

This is called the domain of dependence of the solution at (x_0,t_0). This same idea generalises to quasilinear hyperbolic systems and we shall employ it later.

In conclusion, to illustrate the important notion of an energy integral that arises when working with equations derived from the conservation of physical quantities, let us prove the uniqueness of the solution to the



Domain of dependence

Cauchy problem for slightly generalised two dimensional wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - q(x,y)u - r\,\frac{\partial u}{\partial t}\ , \tag{18}$$

$$\text{with } u(x,y,0) = u_0(x,y),\ \frac{\partial u}{\partial t}(x,y,0) = u_1(x,y)\ , \tag{19}$$

and where we shall assume ρ, k, r to be positive constants and $q(x,y) > 0$. It will be convenient to consider that (18) governs the motion of a membrane with density ρ, tension k per unit length, distributed springing under the membrane with spring constant $q(x,y)$ per unit area and frictional coefficient r.

Then the potential energy within a fixed region R with boundary B of the (x,y)-plane comprises the energy stored in the springing

$$E_s^R(t) = \frac{1}{2}\iint_R qu^2\,dxdy$$

and the energy stored in the membrane

$$E_M^R(t) = -\frac{1}{2}\iint_R uk\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\,dxdy$$
$$+\frac{1}{2}\oint_B uk\,\frac{\partial u}{\partial x}\,ds$$

with $\underline{n}$ the outward drawn unit normal to B and ds a length element of B. The first integral in $E_M^R(t)$ is the negative of the work done by the tension against the interior of R and the second integral the negative of the work done against the boundary.

Green's theorem shows that

$$E_M^R(t) = \frac{1}{2}\iint_R k\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right)\,dx\,dy\ ,$$

so that the total potential energy

$$E_P^R(t) = \frac{1}{2} \iint_R \left\{ k\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + qu^2 \right\} dx\, dy \,. \tag{20}$$

The kinetic energy is

$$E_K^R(t) = \frac{1}{2} \iint_R \rho\left(\frac{\partial u}{\partial t}\right)^2 \; du\, dy \tag{21}$$

so the total energy is

$$E^R(t) = E_K^R(t) + E_P^R(t)$$

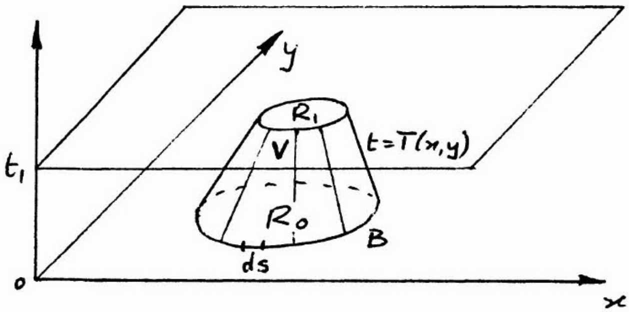
or

$$E^R(t) = \frac{1}{2} \iint_R \left\{ \rho\left(\frac{\partial u}{\partial t}\right)^2 + k\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + qu^2 \right\} dx\, dt \tag{22}$$

It then follows after use of Green's theorem that

$$\frac{d}{dt} E^R(t) = \oint_B \frac{\partial u}{\partial t}\, k\, \frac{\partial u}{\partial n}\, ds - \iint_R r \left(\frac{\partial u}{\partial t}\right)^2 \; dx\, dy \,, \tag{23}$$

which is the outward flux of energy across the boundary and the loss due to friction.



Now let R vary in such a way that at $t = 0$ it is R_0 and at $t = t_1$ it is the smaller domain R_1. The surface between R_0 and R_1, we write in the form $t = T(x,y)$.

Integration of the identity

$$0 = \frac{1}{2}\frac{\partial}{\partial t}\left\{ \rho\left(\frac{\partial u}{\partial t}\right)^2 + k\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + qu^2 \right\}$$

$$- k\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial y}\right)\right] + r\left(\frac{\partial u}{\partial t}\right)^2$$

followed by use of the divergence theorem and some manipulation finally gives the result

$$\iint_{R_0} \frac{1}{2}\left\{\rho\left(\frac{\partial u}{\partial t}\right)^2 + k\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + qu^2\right\} dx\, dy$$

$$= \iiint_V r\left[\frac{\partial u}{\partial t}\right]^2 dt\, dx\, dy + \iint_{\substack{(x,y)\text{ in } R\\ t=T(x,y)}} \frac{1}{2}\left[k\left(\frac{\partial u}{\partial x} + \frac{\partial T}{\partial x}\frac{\partial u}{\partial t}\right)^2\right.$$

$$\left. + k\left(\frac{\partial u}{\partial t} + \frac{\partial T}{\partial y}\frac{\partial u}{\partial t}\right)^2 + \rho\left\{1 - c^2\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2\right]\right\}\left(\frac{\partial u}{\partial t}\right)^2 + qu^2\right] dx\, dy\,, \quad (24)$$

where $c^2 = k/\rho$ and V is the volume of the region concerned.

Now impose on R(t) the condition

$$\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 \leq \frac{1}{c^2}\,. \quad (25)$$

Then all terms on the right-hand side of (24) are non-negative, so if $u = u_t = 0$ at $t = 0$ in R_0 the right-hand side of (24) must vanish, since with zero initial conditions the left-hand side vanishes. In particular this means that u_t must vanish identically on the top and sides of V. However, the top R_1 corresponds to any t_1 so that $u_t \equiv 0$ in V. Since $u = 0$ in R_0 it follows that $u \equiv 0$ in V.

This proves uniqueness, for if two different solutions v and w exist corresponding to the same Cauchy data (19), $u = v-w$ will satisfy the initial data $u = u_t = 0$ at $t = 0$. We have seen this implies $u \equiv 0$ so that $v \equiv w$, and the solution is unique at all points that cannot be reached by a disturbance starting in R_0 and travelling with a speed $\leq c$. The region R_0 now plays the part of the domain of dependence, and the volume V becomes the domain of determinacy.

The limiting case

$$\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 = \frac{1}{c^2}$$

may be interpreted in a useful physical manner if we let $\underline{n}$ be the normal to the ruled surface $t = T(x,y)$. We have

$$\frac{d\underline{n}}{dt} = \frac{1}{|\nabla T|} = \left(\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2\right)^{-\frac{1}{2}} ,$$

so that

$$\frac{d\underline{n}}{dt} = c ,$$

showing that c is the speed of contraction of the region R. The volume V in which the solution is determined by the Cauchy data on R_0 is thus an inverted cone with base R_0.

5. General Effect of Nonlinearity

It is now necessary to make clear that the effect of nonlinearity in a wave equation involves more than the loss of superposibility, for it can also change the entire nature of the solution. This is best shown by a simple non-physical example.

Consider the single first order partial differential equation

$$\frac{\partial u}{\partial t} + f(u)\frac{\partial u}{\partial x} = 0 \tag{26}$$

for the scalar $u(x,t)$ that is subject to the initial condition

$$u(x,0) = g(x) . \tag{27}$$

Now the total differential du is given by

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx ,$$

so that if x and t are constrained to lie on a curve C, then at any point P on C we have

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial t} + \left(\frac{dx}{dt}\right)\frac{\partial u}{\partial x} , \tag{28}$$

where now dx/dt is the gradient of curve C at point P.

Comparison of (26) and (27) now shows that we may interpret (26) as the ordinary differential equation

$$\frac{du}{dt} = 0 \tag{29}$$

along any member of the family of curves C which are the solution curves of

$$\frac{dx}{dt} = f(u) . \qquad (30)$$

These curves C are called the characteristic curves of equation (26). The solution of the partial differential equation (26) has thus been reduced to the solution of the pair of simultaneous ordinary differential equations (29) and (30).

Equation (29) shows that u = const along each of the characteristic curves C. The constant value actually associated with any characteristic curve being equal to the value of u determined by the initial data (27) at the point at which the characteristic curve intersects the initial line t = 0. Setting u = const in (30) then shows that the characteristic curves C of (26) form a family of straight lines. So, if we consider the characteristic through the point $(\xi,0)$ on the initial line, we find after integrating (30) and using (27) that the family of characteristic curves C have the equation

$$x = \xi + tf(g(\xi)) , \qquad (31)$$

where ξ now plays the role of a parameter.

Expressed slightly differently, we have shown that in terms of the parameter $\xi, u = g(\xi)$ at every point of the line (31) in the (x,t) plane. In physical problems t usually denotes time, so that it is then necessary to confine attention to the upper half plane in which $t \geq 0$.

The solution to (26) and (27) may be found in implicit form if ξ is eliminated between $u = g(\xi)$, which is true along a characteristic, and the equation (31) of the characteristic itself. We find the general result

$$u = g(x - tf(u)) . \qquad (32)$$

Result (31) shows that if the functions f and g are such that two characteristics intersect for $t > 0$, then since each one will have associated with it a different constant value of u, it must follow that at such a point the solution will not be unique. This can obviously happen however smooth the two functions may be, since intersection of two characteristics depends merely on the value of $f(g(\xi))$ that is associated with each of the straight line characteristics. This is to say on the two points $(\xi_1,0)$ and $(\xi_2,0)$ of the initial line through which they pass. We conclude from this that such behaviour of solutions is not attributable to any irregularity in the coefficient $f(u)$, or in the initial data $u(x,0) = g(x)$.

Differentiating (32) partially with respect to x gives

$$\frac{\partial u}{\partial x} = \frac{g'(x-tf(u))}{1+tg'(x-tf(u))f'(u)} \tag{33}$$

showing $\partial u/\partial x$ becomes infinite whenever $1 + tg'(x-tf(u))f'(u) = 0$. This is what is often called the gradient catastrophe. In order to extend the solution beyond this point we will need to introduce the concept of a discontinuous solution called a shock. This will be done later.

General References

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Lecture 2. Quasilinear Hyperbolic Systems, Characteristics and Riemann Invariants.

1. Characteristics

The notion of a characteristic curve needs to be introduced in the context of the quasilinear system

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B = 0 , \tag{1}$$

in which U and B are n element column vectors with elements $u_1, u_2, \ldots, u_n$ and $b_1, b_2, \ldots, b_n$, respectively, and A is an n × n matrix with elements a_{ij}. The system (1) will be quasilinear if, in general, the elements a_{ij} of A depend nonlinearly on $u_1, u_2, \ldots, u_n$. When $B \neq 0$ the elements b_i of B may, or may not, depend linearly on $u_1, u_2, \ldots, u_n$. It will be assumed throughout this section that the elements b_i and a_{ij} are *continuous functions* of their arguments.

Although x, t are the natural variables to use when deriving systems of equations describing motion in space and time, they are not necessarily the most appropriate ones from the mathematical point of view. So, as we are interested in the way a solution evolves with time, let us leave the time variable unchanged in system (1), but replace x by some arbitrary curvilinear coordinate ξ and then try to choose ξ in a manner which is convenient for our mathematical arguments. Accordingly, our starting point will be to change from (x, t) to the *arbitrary semi-curvilinear coordinates*

$$\xi = \xi(x, t) , \qquad t' = t . \tag{2}$$

If the Jacobian of the transformation (2) is non-vanishing we may thus transform (1) by the rule

$$\frac{\partial}{\partial t} \equiv \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} \equiv \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial t'}$$

$$\frac{\partial}{\partial x} \equiv \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} \equiv \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi}$$

where, of course, $\partial\xi/\partial t$ and $\partial\xi/\partial x$ are scalar quantities. This leads directly to the transformed equation

$$\frac{\partial U}{\partial t'} + \frac{\partial \xi}{\partial t}\frac{\partial U}{\partial \xi} + \frac{\partial \xi}{\partial x} A \frac{\partial U}{\partial \xi} + B = 0$$

the terms of which may be grouped to yield

$$\frac{\partial U}{\partial t'} + \left(\frac{\partial \xi}{\partial t} I + \frac{\partial \xi}{\partial x} A \right) \frac{\partial U}{\partial \xi} + B = 0 , \qquad (3)$$

where I is the $n \times n$ unit matrix.

Equation (3) may now be considered to be an algebraic relationship connecting the matrix vector derivatives $\partial U/\partial t'$ and $\partial U/\partial \xi$. It is then at once apparent that this equation may only be used to determine $\partial U/\partial \xi$ if the inverse of the coefficient matrix of $\partial U/\partial \xi$ exists. That is to say, if the determinant of the coefficient matrix of $\partial U/\partial \xi$ is non-vanishing. This condition obviously depends on the nature of the curvilinear coordinate lines $\xi(x, t) =$ const., which so far have been chosen arbitrarily. Suppose now that for the particular choice $\xi \equiv \phi$ the determinant does vanish, giving the condition

$$\left| \frac{\partial \phi}{\partial t} I + \frac{\partial \phi}{\partial x} A \right| = 0 . \qquad (4)$$

Then because of this the derivative $\partial U/\partial \phi$ will be indeterminate on the family of lines $\phi =$ const. Consequently, across such lines $\phi(x, t) =$ const., $\partial U/\partial \phi$ may actually be discontinuous. This means that each of the n elements $\partial u_i/\partial \phi$ of $\partial U/\partial \phi$ may be discontinuous across any of the lines $\phi =$ const. To find how, when they occur, these discontinuities in $\partial u_i/\partial \phi$ are related one to the other across a curvilinear coordinate line $\phi =$ const., it is necessary to reconsider equation (3).

We shall now confine attention to solutions U which are *everywhere continuous* but for which the *derivative* $\partial U/\partial \phi$ is *discontinuous* across the particular line $\phi = k$ (say).* Because of the continuity of U, and the continuity of the elements a_{ij} of A and b_i of B, the matrices A and B will experience no discontinuity across $\phi = k$. So, in the neighbourhood of a

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* We call this a weak discontinuity.

typical point P of this line, A and B may be given their actual values at P. In equation (3) there is no indeterminacy of $\partial U/\partial t'$ across the lines ϕ = const., and as $\partial/\partial t'$ denotes differentiation along these lines it must follow that $\partial U/\partial t'$ is everywhere continuous and, in particular, that it is *continuous across the line* $\phi = k$ at P.

Taking these facts into account the differencing equation (3) across the line $\xi \equiv \phi = k$ at P gives

$$\left(\frac{\partial\phi}{\partial t} I + \frac{\partial\phi}{\partial x} A \right)_P \left[\!\left[\frac{\partial U}{\partial \phi} \right]\!\right]_P = 0 , \tag{5}$$

where $[\![\alpha]\!] \equiv \alpha_- - \alpha_+$ signifies the discontinuous jump in the quantity α across the line $\phi = k$, with α_- denoting the value to the immediate left of the line and α_+ the value to the immediate right at P. As the point P was any point on this line the suffix P may now be omitted. The operator $\partial/\partial\phi$ is *differentiation normal to the curves* ϕ = const., so that equations (5) express compatibility conditions to be satisfied by the component of the derivative of U on either side of and normal to these curves in the (x, t)-plane.

This is a homogeneous system of equations for the n jump quantities $[\![\partial u_i/\partial\phi]\!] \equiv (\partial u_i/\partial\phi)_- - (\partial u_i/\partial\phi)_+$ and there will only be a non-trivial solution if the determinant of the coefficients vanishes. The condition for this is

$$\left| \frac{\partial\phi}{\partial t} I + \frac{\partial\phi}{\partial x} A \right| = 0 . \tag{6}$$

However, along the lines ϕ = const. we have, by differentiation,

$$\frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x}\frac{dx}{dt} = 0$$

so that these lines have the gradient

$$\frac{dx}{dt} = - \frac{\partial\phi}{\partial t} \Bigg/ \frac{\partial\phi}{\partial x} \equiv \lambda \quad \text{(say)}. \tag{7}$$

Combining (6) and (7) we deduce that λ must be such that

$$| A - \lambda I | = 0 . \tag{8}$$

Consequently the λ in (7) can only be one of the eigenvalues of A, and since (5) can be re-written

$$(A - \lambda I) \left[\!\left[\frac{\partial U}{\partial \phi} \right]\!\right] = 0 , \qquad (9)$$

the column vector $[\![\partial U/\partial \phi]\!]$ must be proportional to the corresponding right eigenvector of A. This, then, determines the ratios between the n elements $[\![\partial u_i/\partial \phi]\!]$ of the vector $[\![\partial U/\partial \phi]\!]$ that we were seeking.

As A is an n × n matrix it will have n eigenvalues. If these are real and distinct, integration of equations (7) will give rise to n distinct families of real curves $C^{(1)}, C^{(2)}, \ldots, C^{(n)}$ in the (x, t)-plane:

$$C^{(i)} : \frac{dx}{dt} = \lambda^{(i)} , \qquad i = 1, 2, \ldots, n. \qquad (10)$$

If x denotes a distance and t a time, the eigenvalues will have the dimensions of a speed. Any one of these families of curves $C^{(i)}$ may be taken for our curvilinear coordinate lines ϕ = const. The $\lambda^{(i)}$ associated with each family will then be the speed of propagation of the matrix column vector $[\![\partial U/\partial \phi]\!]$ along the curves $C^{(i)}$ belonging to that family.

When the eigenvalues $\lambda^{(i)}$ of A are all real and distinct, so that the propagation speeds are also all real and distinct, and there are n distinct linearly independent right eigenvectors $r^{(i)}$ of A satisfying the defining relation

$$Ar^{(i)} = \lambda^{(i)} r^{(i)}, \qquad \text{for } i = 1, 2, \ldots, n, \qquad (11)$$

the system of equations (1) will be said to be <u>totally hyperbolic</u>. We may, if we desire, replace the words right eigenvector by left eigenvector in this definition, where the left eigenvectors I of A satisfy the defining relation

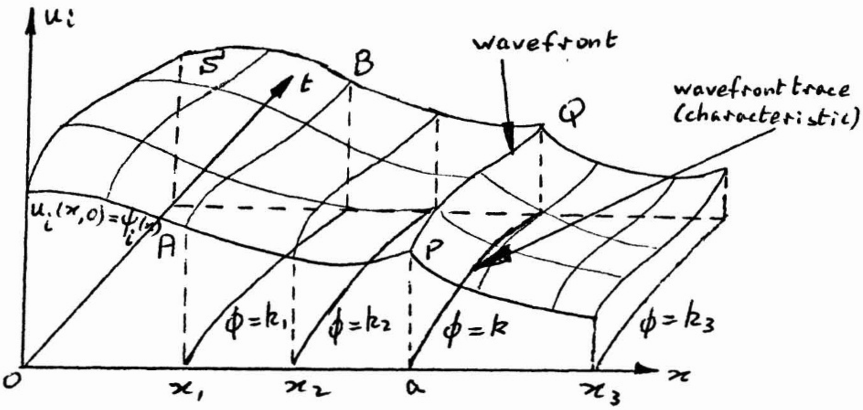
$$I^{(i)} A = \lambda^{(i)} I^{(i)}, \qquad \text{for } i = 1, 2, \ldots, n. \qquad (12)$$

The families of Curves $C^{(i)}$ defined by integration of equations (10) are called the families of <u>characteristic curves</u> of system (1).

The relationship between characteristic curves and the solution vector U to system (1) is illustrated in the Figure in the case of a typical element u_i of U. Here it has been assumed that initial conditions have been specified for system (1) in the form

$$U(x, 0) = \Psi(x) ,$$

where the ith element u_i of U has for its initial condition $u_i(x, 0) = \psi_i(x)$.



Since it was not necessary that $\partial U/\partial\phi$ should be discontinuous across the characteristics ϕ = const., it must follow that continuous and differentiable elements of the initial data $u_i(x, 0) = \psi_i(x)$ will also propagate along characteristics. In the case of the element of initial data at A, this will propagate along the characteristic $\phi = k_1$ (say) starting from the point $(x_1, 0)$ which is the projection of A onto the initial line. The characteristic $\phi = k_1$ is then the projection onto the (x, t)-plane of the path AB followed by the element of the solution surface S that started at A. Characteristics corresponding to $k = k_2, k_3, k_4$, etc., may be interpreted in similar fashion.

To distinguish between initial and boundary value problems it is necessary to classify arcs Γ in the (x, t)-plane as being either timelike or spacelike. This is done by assigning to each characteristic arc an arrow showing the direction corresponding to increasing t, and then