Logica Universalis

Towards a General Theory of Logic

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Editor

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Preface

Logica Universalis (or *Universal Logic*, *Logique Universelle*, *Universelle Logik*, in vernacular languages) is not a new logic, but a general theory of logics, considered as mathematical structures. The name was introduced about ten years ago, but the subject is as old as the beginning of modern logic: Alfred Tarski and other Polish logicians such as Adolf Lindenbaum developed a general theory of logics at the end of the 1920s based on consequence operations and logical matrices. Talking about the papers of Tarski dealing with this topic, John Etchemendy says: "What is most striking about these early papers, especially against their historical backdrop, is the extraordinary generality and abstractness of the perspective adopted" [4]. After the second world war, this line of work was pursued mainly in Poland and became a bit of an esoteric subject. Jerzy Łoś's fundamental monograph on logical matrices was never translated in English and the work of Roman Suszko on abstract logics remained unknown outside of Poland during many years.

Things started to change during the 1980s. Logic, which had been dominated during many years by some problems related to the foundations of mathematics or other metaphysical questions, was back to reality. Under the impulsion of artificial intelligence, computer science and cognitive sciences, new logical systems were created to give an account to the variety of reasonings of everyday life and to build machines, robots, programs that can act efficiently in difficult situations, for example that can smoothly process inconsistent and incomplete information. John McCarthy launched non-monotonic logic, few years later Jean-Yves Girard gave birth to linear logic. Logics were proliferating: each day a new logic was born. By the mid eighties, there were more logics on earth than atoms in the universe. People began to develop general tools for a systematic study of this huge amount of logics, trying to put some order in this chaotic multiplicity. Old tools such as consequence operations, logical matrices, sequent calculus, Kripke structures, were revived and reshaped to meet this new goal. For example sequent calculus was the unifying instrument for substructural logics. New powerful tools were also activated, such as labelled deductive systems by Dov Gabbay.

Amazingly, many different people in many different places around the world, quite independently, started to work in this new perspective of a general theory of logics, writing different monographs, each one presenting his own way to treat the problem: Norman Martin's emphasis was on Hilbert systems [9], Richard Epstein's,

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on semantical tools, in particular relational structures and logical matrices [5], Newton da Costa's, on non truth-functional bivalent semantics [7], John Cleave's, on consequence and algebra [3], Arnold Koslow's, on Hertz's abstract deductive systems [8]. This was also the time when was published a monograph by Ryszard Wójcicki on consequence operations making available for the first time to a wide public the main concepts and results of Polish logic [10], and the time when Dov Gabbay edited a book entitled *What is a logical system?* gathering a collection of papers trying to answer this question in many different ways [6]. Through all these publications, the generality and abstractness of Tarski's early work was being recovered. It is surrounded by this atmosphere that I was doing my PhD [2] and that I coined in the middle of a winter in Poland the expression "universal logic" [1], by analogy to the expression "universal algebra".

The present book contains recent works on universal logic by first-class researchers from all around the world. The book is full of new and challenging ideas that will guide the future of this exciting subject. It will be of interest for people who want to better understand what logic is. It will help those who are lost in the jungle of heterogeneous logical systems to find a way. Tools and concepts are provided here for those who want to study classes of already existing logics or want to design and build new ones.

In Part I, different frameworks for a general theory of logics are presented. Algebra, topology, category theory are involved. The first paper, written by myself, is a historical overview of the different logical structures and methods which were proposed during the XXth century: Tarski's consequence operator and its variants in particular Suszko's abstract logic, structures arising from Hertz and Gentzen's deductive systems, da Costa's theory of valuation, etc. This survey paper presents and explains many concepts that are used in other papers of the book. The following paper, by Marta García-Matos and Jouko Väänänen, gives a hint of how *abstract model theory* can be used for developing universal logic. Although abstract logic and abstract model theory are expressions which look similar, they refer to two different traditions. Abstract logic has been developed by Suszko in the context of the Polish tradition focusing on a general theory of zero-order logics (i.e. propositional logics). On the other hand, the aim of abstract model theory has been the study of classes of higher order logics. The combination of abstract model theory with abstract logic is surely an important step towards the development of universal logic. It is also something more than natural if we think that both theories have their origins in the work of Alfred Tarksi. Steffen Lewitzka's approach is also model-theoretical, but based on topology. He defines in a topological way logic-homomorphims between abstract logics, which are mappings that preserve structural properties of logics. And he shows that those model-theoretical abstract logics together with a strong form of logic-homomorphisms give rise to the notion of institution. Then comes the work of Ramon Jansana which is a typical example of what is nowadays called *abstract algebraic logic*, the study of algebraization of logics, a speciality of the Barcelona logic group. Within this framework, abstract logics are considered as generalized matrices and are used as models for logics. Finally, Pierre Ageron's paper deals with logics for which the law of *self-deductibility* does not hold. According to this law, a formula is always a consequence of itself, it was one of the basic axiom of Tarski's consequence operator. Ageron shows here how to develop logical structures without this law using tools from category theory.

The papers of Part II deal with a central problem of universal logic: the question of identity between logical structures. A logic, like classical logic, is not a given structure, but a class of structures that can be identified with the help of a given criterion. According to this criterion, we say that structures of a given class are equivalent, congruent or simply identical. Although this question may at first looks trivial, it is in fact a very difficult question which is strongly connected to the question of what a logical structure is. In other words, it is not possible to try to explain how to identify different logical structures without investigating at the same time the very nature of logical structures. This is what makes the subject deep and fascinating. Three papers and seven authors are tackling here the problem, using different strategies. Caleiro and Gonçalves's work is based on concepts from category theory and they say that two logics are the same, *equipollent* in their terminology, when there exist uniform translations between the two logical languages that induce an isomorphism on the corresponding theory spaces. They gave several significative illustrations of equipollent and non equipollent logics. Mossakowski, Goguen, Diaconescu and Tarlecki use also category theory, more specifically their work is based on the notion of *institution.* They argue that every plausible notion of equivalence of logics can be formalized using this notion. Lutz Straßburger's paper is proof-theoretically oriented, he defines identity of proofs via *proof nets* and identity of logics via pre-orders.

In part III, different tools and concepts are presented that can be useful for the study of logics. The papers by Arnon Avron and by Carlos Caleiro and co. both deal with a concept very popular in the Polish tradition, the concept of logical matrices, the basic tool for many-valued logics. In his paper Avron studies the notion of *non-deterministic matrices* which allows to easily construct semantics for proof systems and can be used to prove decidability. This tool can be applied to a wide range of logics, in particular to logics with a formal consistency operator. Caleiro, Carnielli, Coniglio and Marcos discuss Suszko's thesis, according to which any logic is bivalent, and present some techniques which permit to construct in a effective way a *bivalent semantics*, generally not truth-functional, from a many-valued matrix. Their paper is illustrated by some interesting examples, including Belnap's four-valued logic. Then comes a paper by David Makinson, one of the main responsible for the revival of Tarski's consequence operator at the beginning of the 1980s. He used it at the main tool, on the one hand for the development together with Carlos Alchourrón and Peter Gärdenfors, of theory change (universally known today under the acronym AGM), on the other hand as a basis for a general theory of non monotonic logics. In both cases, Makinson's use of Tarski's theory was creative, he kept the original elegant abstract spirit, but widened and extended the basic underlying concepts. Here again he is innovative defining within classical propositional logic two new concepts, *logical friendliness and sympathy*, which lead to some consequence relations with non standard properties. The paper by Lloyd Humberstone is no less original and brilliant, he studies the very interesting phenomenon of *logical discrimination*. The question he examines is in which circumstances, discrimination, i.e. distinction between formulas, is correlated with the strength of a logic. The work of Humberstone is a very good example of the philosophical import of universal logic. By a careful examination of a phenomenon like discrimination, that requires a precise mathematical framework, one can see to which extent a statement with philosophical flavor saying that discrimination is inversely proportional to strength is true or not.

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Part I

Universal Logic: Frameworks and Structures

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From Consequence Operator to Universal Logic: A Survey of General Abstract Logic

Jean-Yves Beziau

Abstract. We present an overview of the different frameworks and structures that have been proposed during the last century in order to develop a general theory of logics. This includes Tarski's consequence operator, logical matrices, Hertz's Satzsysteme, Gentzen's sequent calculus, Suszko's abstract logic, algebraic logic, da Costa's theory of valuation and universal logic itself.

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1. Introduction

During the XXth century, numerous logics have been created: intuitionistic logic, quantum logic, modal logic, many-valued logic, relevant logic, paraconsistent logic, erotetic logic, polar logic, linear logic, non-monotonic logic, dynamic logic, free logic, fuzzy logic, paracomplete logic, etc. And the future will see the birth of many other logics that one can hardly imagine at the present time.

Facing this incredible multiplicity, one can wonder if there is not a way to find common features which allow one to unify the study of all these particular systems into a science called *logic*.

In what follows we describe various attempts that have been made during the XXth century to develop a general theory of logics.

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2. Tarski's consequence operator

2.1. Tarski's three axioms

Undoubtedly, Tarski has, among many other things, to be considered as the initiator of a general theory of logics.

At the end of the twenties, he launched the theory of consequence operator [43]. This theory is about an "operator", a function Cn defined on the power set of a given set $\mathbb{S}$. Following the philosophical ideas of his master, Leśniewski, Tarski calls these objects "meaningful sentences". But in fact, the name does not matter, the important thing is that here Tarski is considering a very general theory, because the *nature* of these objects is not specified. For Tarski, these sentences can be sentences of any kind of scientific languages, since his work is concerned with the *methodology of deductive sciences*, and not only with *metamathematics*. The function Cn obeys three basic axioms, for any *theories* (i.e. sets of sentences) T and U:

[TAR1] $T \subseteq CnT$

[TAR2] if $T \subseteq U$ then $CnT \subseteq CnU$

[TAR3] $CnCnT \subseteq CnT$

Hereafter, a structure $\langle \mathbb{S}; Cn \rangle$ where Cn obeys the three above axioms will be called a *Tarski structure.* [1]

2.2. Axiomatizing axiomatic proof systems

Why these axioms? Tarski wanted to give a general characterization of the notion of *deduction.* At this time, the standard notion of deduction was the one given by what is called nowadays *Hilbert-type proof systems*, or *axiomatic proof systems.* It is easy to check that any notion of deduction defined with the help of this kind of systems obeys the three above axioms.

One could say that Tarski was in this sense axiomatizing axiomatic proof systems. It is very important however to understand the difference between the two occurrences of the word *axiom* here. Tarski's axioms are not axioms of a proof system, although they can be considered as such, at a more complex level. One should rather consider these axioms model-theoretically, as defining a certain class of structures.

One can wonder if these three axioms characterize exactly the notion of deduction in the sense that any Tarski structure $\langle \mathbb{S}; Cn \rangle$ verifying these axioms can be defined in a Hilbertian proof-theoretically way.

[1] In this paper we will do a bit of taxonomy, fixing names to different logical structures.

2.3. Semantical consequence and completeness

Before examining this question, let us note that these axioms axiomatize also the *semantical* or *model-theoretical notion of consequence* given later by Tarski [45]. We do not know if Tarski had also this notion in the back of his mind when he proposed the consequence operator.

Anyway, there is an interesting manner to connect these two notions which forms the heart of a general completeness theorem. If we consider a Tarski structure $\langle \mathbb{S}; Cn \rangle$ and the class of *closed theories* of this structure, i.e., theories such that $CnT = T$, this class forms a sound and complete semantics for this structure, in the sense that the semantical notion of consequence $\models$ defined by:

$$T \models a \text{ iff for any closed theory } U \text{ such that } T \subseteq U,\ a \in U$$

coincides with Cn.

Now back to the previous point, it is clear that Tarski's three axioms do not characterize the notion of Hilbertian proof-theoretical deduction, since for example there are some structures $\langle \mathbb{S}; Cn \rangle$ like second-order logic that obey these three axioms but cannot be defined in this proof-theoretical sense. This is because second-order logic is not compact, or more precisely, finite.

2.4. Compactness and finiteness

Tarski had also an *axiom for compactness*:

[COM] $CnT = \bigcup CnF$ ($F \subseteq T$, F finite)

In classical logic, this axiom is equivalent to the *axiom of finiteness*:

[FIN] if $a \in CnT$ then $a \in CnF$ for a finite $F \subseteq T$.

But in general they are not. Clearly it is the axiom of finiteness which characterizes the Hilbertian proof-theoretical notion of deduction.

Once one has this axiom, one has also a more interesting semantical notion of consequence. Let us call *maximal* a theory T, such that $CnU = \mathbb{S}$, for every strict extension U of T and *relatively maximal* a theory T such that there is a a such that $a \notin T$ and for every strict extension U of T, $a \in CnU$. The class of relatively maximal theories characterizes any finite consequence operator (i.e. a consequence operator obeying the finiteness axiom together with Tarski's three axioms). In the case of an *absolute* Tarski structure, i.e. a structure where all relatively maximal theories are also maximal [2], maximal theories characterize finite consequence operators, but it is not true in general [9].

[2]The terminology "absolute" was suggested by David Makinson.

3. Hertz and Gentzen's proof systems

3.1. Hertz's Satzsysteme

It is difficult to know exactly the origin of the work of Paul Hertz about *Satzsysteme* developed during the 1920s [31]. But something is clear, it emerged within the Hilbertian stream and it is proof-theoretically oriented, although it is a very abstract approach ; like in the case of Tarski's consequence operator, the nature of the basic objects is not specified.

What Hertz calls a *Satz* is something of the form $u_1 \dots u_n \rightarrow v$. One could interpret this as the sentence "u_1 and...and u_n implies v", considering $\rightarrow$ as material implication, and Hertz himself suggests this, saying that $\rightarrow$ is taken from Whitehead and Russell's *Principia Mathematica*.

However, we shall interpret here Hertz's *Satz* $u_1 \dots u_n \rightarrow v$ in the perspective of Gentzen's work considering it as the prototype of a Gentzen's sequent and we shall just call such a *Satz*, a protosequent. "u" and "v" are called "elements" in Hertz's terminology. Following again Gentzen, we will just consider that they are sentences of a possible unspecified language. Hertz uses the word "complex" to denote a finite site Γ of "elements", we will just call such a set a finite theory.

Hertz's notion of *Satzsysteme* is based on a notion of proof which is similar to the Hilbert's one, except that elements of the proof are protosequents. Therefore rules in *Satzsysteme* have as premises, protosequents, and as a conclusion a protosequent. Axioms are protosequents. There are only one kind of axiom and two rules in Hertz's system:

[HER1] $\Gamma \rightarrow \alpha \ (\alpha \in \Gamma)$

[HER2] $\dfrac{\Gamma \rightarrow \alpha}{\Gamma\Delta \rightarrow \alpha}$

[HER3] $\dfrac{\Gamma \rightarrow \alpha \qquad \Delta\alpha \rightarrow \beta}{\Gamma\Delta \rightarrow \beta}$

If we consider, something that Hertz didn't, the structure $\langle \mathbb{S}; \vdash \rangle$ generated by such a system in the following way: $T \vdash a$ iff there exists a finite subtheory Γ of T such that there is a proof of the protosequent $\Gamma \rightarrow a$ in the system, we have a structure which is equivalent, modulo trivial exchanges between $\vdash$ and Cn, to a Tarski structure.

As we have seen, Tarski's motivations are clear and one can perceive the interest of his proposal. In the case of Hertz's work, it is not clear at all. One can see a step towards a kind of generalization of Hilbert-type proof-theoretical concepts. But at first, Hertz's notion of *Satz* is quite strange. In the light of Gentzen's work, we are now conscious of the incredible power of this notion, but many people think that without Gentzen's work, Hertz's work would have been completely ignored.

In fact Hertz's work is generally not well-known. It is therefore important to stress that Gentzen started his researches, which will lead him to his famous sequent calculus, by studying Hertz's work, probably on a suggestion of Paul Bernays. Gentzen's first paper [29] is entirely devoted to Hertz's system and among several results he proves that this system is sound and complete with respect to the semantics of closed theories (although he doesn't use such a language). Due to his paper, the model-theoretical notion of consequence and this general related completeness theorem can be credited to Gentzen as well as to Tarski. Apparently his work was carried out in total independence to the work of Tarski.

3.2. Gentzen's sequent calculus

Gentzen's sequent calculus differs in several points to Hertz's system. Instead of Hertz's Satz, Gentzen considers "sequents", i.e, objects of the form $u_1, \ldots, u_n \to v_1, \ldots, v_n$ where $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are sequences of sentences (mind the comma!), hence the name "sequent calculus". Gentzen's rules are divided in two categories: structural rules and logical rules. Logical rules are rules concerning logical operators. Such rules appear here because Gentzen is not only interested to work at the "abstract" level but also with specific logics, mainly classical and intuitionistic logics. Gentzen's structural rules are the following:

[GEN1] $\Sigma \to \alpha \ (\alpha \in \Sigma)$

[GEN2l] $\dfrac{\Sigma \longrightarrow \Xi}{\alpha, \Sigma \longrightarrow \Xi}$ [GEN2r] $\dfrac{\Sigma \longrightarrow \Xi}{\Sigma \longrightarrow \Xi, \alpha}$

[GEN3] $\dfrac{\Sigma \longrightarrow \Xi, \alpha \qquad \alpha, \Omega \longrightarrow \Pi}{\Sigma, \Omega \longrightarrow \Xi, \Pi}$

[GEN4l] $\dfrac{\Sigma(\alpha, \alpha) \longrightarrow \Xi}{\Sigma(\alpha) \longrightarrow \Xi}$ [GEN4r] $\dfrac{\Sigma \longrightarrow \Xi(\alpha, \alpha)}{\Sigma \longrightarrow \Xi(\alpha)}$

[GEN5l] $\dfrac{\Sigma(\alpha, \beta) \longrightarrow \Xi}{\Sigma(\beta, \alpha) \longrightarrow \Xi}$ [GEN5r] $\dfrac{\Sigma \longrightarrow \Xi(\alpha, \beta)}{\Sigma \longrightarrow \Xi(\beta, \alpha)}$

where Σ, Ω, Π and Ξ are sequences and something like $\Sigma(\alpha, \beta)$ means that α and β are occurrences of formulas appearing in the sequence Σ in that order.

[GEN1], [GEN2] and [GEN3] are adaptations of [HER1], [HER2] and [HER3] to the sequent context. The rules [GEN4] and [GEN5] were implicit in the case of Hertz's system, but if one considers sequents instead of finite sets of sentences, these rules are necessary.

Later on, people started to work with Tait's version of Gentzen's sequent calculus. It is a Hertzianization of Gentzen's system where finite sets are considered instead of sequences, and where therefore there are no contraction [GEN4] and permutation [GEN5] rules.

In view of Tait's version, one may think that Gentzen's system is a useless *détour*. But it is not, as the recent development of substructural logics has dramatically shown.

3.3. Scott structure

An important point is that Gentzen is considering a multiplicity of sentences on the right. So even if one considers Tait's versions of Gentzen's structural rules, we have here something different from Hertz's rules, in particular we must have two thinning rules, [GEN2l] and [GEN2r].

The multiplicity of the sentences on the right is very important in Gentzen's original system, since, as it is known, if one reduces this multiplicity to unity on the right, one goes from classical logic to intuitionistic logic.

However, even if we stay at the abstract level, the multiplicity is an important thing that permits to work with more symmetry. The structure generated by an abstract (only structural rules) Tait's version of Gentzen's sequent calculus is a structure of type $\langle \mathbb{S}; \bowtie \rangle$ where $\bowtie$ is a relation on $\mathcal{P}(\mathbb{S}) X \mathcal{P}(\mathbb{S})$.

This relation obeys a straightforward generalization of the three Tarskian axioms. We will call such a structure, a Scott structure, since Dana Scott made important contribution working with this kind of structures, generalizing for example Lindenbaum theorem for them (see [38]). This kind of approach is usually known under the banner "multiple-conclusion logic" (see [39]).

3.4. Substructural structure

A substructural structure $\langle G; \bowtie \rangle$ is a Scott structure where a magma $G = \langle \mathbb{S}; * \rangle$ is considered instead of the naked set $\mathbb{S}$. A magma is just a set with a binary operation $*$.[3] Some specific axioms can be added for the operation *. Gentzen's notion of sequents can be designed in this way, and therefore substructural structures are a refinement of Gentzen's idea. Gentzen's notion of sequents is quite precise but for example associativity is an implicit supposition of it. Considering a magma, one can turn this hypothesis explicit, with an axiom of associativity for $*$, or withdraw it and work with non associativity. In a substructural structure in general there are also no specific axioms for $\bowtie$.

In the last twenty years the amazing development of linear logic [30] and non monotonic logics has shown the fundamental role of substructural structures (see [28], [37]).

3.5. Turning style

In the context of Gentzen's sequent calculus, the Hertz-Gentzenian symbol $\rightarrow$ is very often replaced by the turnstile $\vdash$ (in particular due to the fact that people now use $\rightarrow$ for material implication instead of the old $\supset$).

This change of symbol seems harmless, but in fact one has to be very careful, because it leads to a confusion between a proof-theoretical system with axioms and rules with the structure generated by this system. People like Scott and those working with substructural logics generally do not make this difference, and there is a tendency to use the same name for Gentzen's structural rules and for axioms applying to the relation $\bowtie$.

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[3] The terminology "magma" is due to Charles Ehresmann.

For example the axiom stating the transitivity of $\bowtie$ is sometimes called cut. This can lead to serious misunderstandings of the cut-elimination theorem. Gentzen's system for classical logic with cut is equivalent to his system without cut, but the cut rule is not derivable in the cut-free system, although this system generates a transitive relation $\bowtie$, since the two systems are equivalent (see [8]).

Another example is the confusion between thinning rules [GEN2] and the axiom of monotonicity [TAR2]. A proof system can have no thinning rules and be monotonic.

4. Matrix theory and abstract logic

4.1. Łukasiewicz and Tarski's concept of logical matrices

Influenced by Tarski's theory of consequence operator, Polish logicians have developed since the 1930s a general theory of zero-order logics (i.e. propositional or sentential logics). This kind of stuff is generally known under the name Polish logic (see [12]). A central concept of Polish logic is the notion of matrix, or logical matrix. In fact one could say that Polish logic is the fruit of the wedding between the concepts of consequence operator and logical matrix. Polish logicians have not developed the theory of consequence operator by itself, at the abstract level, maybe because they thought it was sterile. Anyway they have shown that its combination with matrix theory is highly fruitful.

The concept of logical matrix was introduced in Poland by Łukasiewicz, through the creation of many-valued logic. However it is Tarski who saw the possibility of using this theory as a basic tool for a systematic study of logics. It is clear that matrix theory does not reduce to many-valued logics, as shown by its use for the proof of independence of axioms of the two-valued propositional logics. Matrices are models of zero-order non classical logics. In fact the consideration of *models of zero-order non classical logics* led Tarski to *classical first-order model theory* (see [38]).

4.2. Lindenbaum's matrix theorem

The first important general result about matrices is due to Lindenbaum. A matrix $\mathcal{M}$ is an algebra $\mathcal{A} = \langle \mathbb{A}; f \rangle$ together with a subset $\mathbb{D}$ of $\mathbb{A}$, whose elements are called designated values. When one uses logical matrices, one considers the set $\mathbb{S}$ of sentences of a logical structure as an algebra, an absolutely free algebra (explicit consideration of this fact is also credited to Lindenbaum). Operators of this algebra represent zero-order connectives.

Let us call a *Lindenbaum structure*, a structure $\langle \mathcal{S}; \mathbb{T} \rangle$ where $\mathcal{S}$ is an absolutely free algebra of domain $\mathbb{S}$ and $\mathbb{T}$ is a subset of $\mathbb{S}$. One can wonder if it is possible to find a logical matrix $\mathcal{M}$ which characterizes this structure in the sense that any homomorphism η from $\mathcal{S}$ to the algebra $\mathcal{A}$ of the matrix is such that:

$$\alpha \in \mathbb{T} \text{ iff } \eta(\alpha) \in \mathbb{D}$$

There are many logics, such as some modal logics like S5 or intuitionistic logic, that cannot be characterized by finite matrices (i.e. matrices where the domain of the algebra is finite). However Lindenbaum has shown that every Lindenbaum structure stable under substitution can be characterized by a matrix of cardinality superior or equal to the cardinality of the language (i.e. the domain of the structure).

Lindenbaum was killed during the second world war, but just after the war his work was disseminated in Poland through the monograph of Jerzy Łoś entirely devoted to logical matrices (see [32]). Lindenbaum's theorem was generalized by Wójcicki for the case of Łoś structures. A Łoś structure is a structure of type $\langle \mathcal{S}; Cn \rangle$ where S is an absolutely free algebra and Cn a structural consequence operator, i.e. a consequence operator obeying the three basic Tarski's axioms and the following condition:

$$\text{for every endomorphism } \epsilon \text{ of } \mathcal{S},\ \epsilon CnT \subseteq Cn\epsilon T$$

In other words, this means that Cn is stable under substitution. This crucial notion was introduced in [33].

Matrix theory was also applied to Scott structures by Zygmunt (see [48]).

4.3. Suszko's abstract logic

Suszko and his collaborators have shown that all known logics are structural. Later on, Suszko developed a general study of logics that he called "abstract logic" considering as basic structure a Suszko structure, i.e. a structure of type $\langle \mathcal{A}; Cn \rangle$ where $\mathcal{A}$ is an abstract algebra and Cn a consequence operator obeying the three basic Tarski's axioms (see [18]).

Abstract logic in this sense is very close to universal algebra. Concepts of category theory and model theory can also be fruitfully applied for its development. With Suszko's abstract logic, the general theory of logics reached the level of mathematical maturity, turning really into a mathematical theory in the modern sense of the word.

One could say: "very well, this is mathematics, but this is not *about* mathematics!", since an abstract logic is a model for a propositional logic and we all know that such a logic, be it classical, intuitionistic or whatever, is not rich enough to fully represent mathematical reasoning.

I have proposed to generalize the notion of abstract logic considering structures of type $\langle \mathcal{A}; Cn \rangle$ where $\mathcal{A}$ can be an infinitary algebra, in order to represent logics of order superior to zero, taking in account the fact that higher order languages can be described by infinitary algebras (see [3]).

5. Algebraic logic

5.1. Logic and algebra

Algebraic logic is an ambiguous expression which can mean several things. One could think that it is crystal clear and that algebraic logic means the study of logic from an algebraic point of view. But this is itself ambiguous, because this in turn means two things:

(1) The study of logic using algebraic tools

(2) Logics considered as algebraic structures.

(2) implies (1) but not necessarily the converse. It is clear that when one considers a logic as a Lindenbaum structure or a Łoś structure and considers the problem of characteristic matrices, this involves mainly algebraic concepts. One can even say that these structures as well as Suszko's abstract logics are algebraic structures. Roughly speaking this is right. But if one wants to be more precise, it is important to emphasize two points ; this will be the subject of the two next subsections.

5.2. Cross structures

First these structures are not exactly algebras, according to *Birkhoff's standard definition of algebra*. To call these structures algebras leads to a general confusion according to which any mathematical structure is called an algebra. In fact an abstract logic in the sense of Suszko is a mixture of topological concepts and algebraic concepts. Algebraic concepts are related to the structure of the language - algebraic operators representing logical operators - and topological concepts are related to the consequence operator Cn. In fact Tarski was probably influenced by topology when he developed the theory of consequence operator since topology was very popular at this time in Poland and Tarski himself was collaborating with Kuratowski. Łoś and Suszko structures are in fact cross structures according to Bourbaki's terminology, they are the result of crossing two fundamental mother structures: topological and algebraic structures.

To call "algebraic logic" a general theory of logics involving algebraic concepts is misleading. Polish logic, which is such a theory, is often assimilated with algebraic logic, by opposition to a more traditional approach to logic based on intuitive concepts related to linguistics. But when the people, following this second approach, say that Polish logic is algebraic logic, they simply identify algebra with mathematics, or in the best case algebraic structures with mathematical structures.

5.3. Lindenbaum-Tarski algebras

It is not rare to hear that classical propositional logic is a boolean algebra. Tarski at the beginning of the 1930s showed how to reduce classical logic to a boolean algebra by factorizing the structure (cf. [44]). The factorized structure is called a *Lindenbaum-Tarski algebra*, LT-algebra for short. The concept of LT-algebra was then extended to other logics. Algebraic logic in this sense is the study of logics, via their LT-algebras, and more generally the study of algebras which can be considered as LT-algebras of some logics. This means in general that people are considering algebraic structures of type $\langle \mathcal{A}; \leqslant \rangle$ where $\leqslant$ is an order relation and $\mathcal{A}$ is an algebra whose operators have intended logical meaning: conjunction, disjunction, implication and negation. This is for example the case of the famous

Birkhoff-von Neumann's quantum logic which is in fact an algebraic structure of this kind. These structures are tightly linked with lattices. A general study of these structures has been developed by H.B.Curry (see [26]).

If one considers algebraic logic as the study of this kind of structures and this is probably the only rigorous way to use this terminology, it seems then that algebraic logic is too restricted for developing a general theory of logics. Firstly because the notion of logical consequence cannot be properly represented by an order relation $\leqslant$, one has to consider at least a consequence relation or a binary Scott-type relation; secondly because there are logics which cannot properly be handle through LT-algebras. This is the case of simple logics, logics that have no non-trivial congruence relations and which cannot be factorized, like da Costa's paraconsistent logic C1 (see [5]).

However these last twenty years work in algebraic logic has made important advances through the introduction of several new concepts such as protoalgebraization and the correlated refinement of LT-algebra (see [27], [17]).

6. Da Costa's theory of valuation

6.1. Every logic is two-valued

As we have seen, closed theories form a sound and complete semantics for any Tarski structure and relatively maximal theories form a sound and complete semantics for Tarski structures obeying the finiteness axiom. Now instead of considering theories, one can consider the characteristic functions of these theories, these are *bivaluations*.

The above results can therefore be reinterpreted as saying that every Tarski structure has a bivalent semantics, and they justify, at least if we restrict ourselves to such structures, a general theory of logics based on the concept of bivaluations. Newton da Costa's theory of valuation is such a theory.

The advantage of such a theory is that it is based on the semantical intuitive ideas of true and false and that it can be seen as a natural generalization of the bivalent semantics for classical propositional logic, which can be applied to non-classical logics and high-order logics.

6.2. Bivalency and truth-functionality

This generalization is however not so natural in the sense that one central feature of the semantics of classical propositional logic is lost in most of the cases: truth-functionality. Therefore the theory of valuation is mainly a theory of non truth-functional bivalent semantics. An interesting result due to da Costa shows that truth-functional bivalent semantics determine only logics which are sublogics of classical logic (see [23]).

Semantics of bivaluations can be developed for many-valued logics, such as Łukasiewicz's three-valued logic. But they are more interesting for logics which are not truth-functional, in the sense that they cannot be characterized by a finite matrix. In fact, originally da Costa built semantics of valuation for his paraconsistent logic C1, which is a non truth-functional logic [22].

6.3. Bivaluations, truth-tables and sequent calculus

Despite of the non truth-functionality of such semantics, it is possible to construct truth-tables which are quite similar to the classical ones, and which provide decision methods.

I have linked the theory of valuations with sequent calculus showing how it is possible to translate conditions defining bivaluations into sequent rules and vice-versa. Combining action of valuations upon sequent rules, in the spirit of Gentzen's 1932 proof [29], with Lindenbaum-Asser theorem, I have given a general version of the completeness theorem, from which it is possible to derive instantaneously many specific completeness theorems (see [2], [11]),

7. Universal logic

7.1. Universality and trivialization

Wójcicki said once that his objective was to trivialize the completeness theorem. What does this mean? It means finding a general formulation of this theorem from which particular theorems appear as trivial corollaries.

In a proof of a completeness theorem for a given logic, one may distinguish the elements of the proof that depend on the specificity of this logic and the elements that do not depend on this peculiarity, that we can call *universal*.

This distinction is important from a methodological, philosophical and mathematical point of view. The first proofs of completeness for propositional classical logic give the idea that this theorem is depending very much on classical features. Even one still gets this impression with recent proofs where the theorem is presented using the concept of maximal consistent set which seems to depend on classical negation. In fact this idea is totally wrong and one can present the completeness theorem for classical propositional logic in such a way that the specific part of the proof is trivial, i.e. one can trivialize the completeness theorem.

One central aim of a general theory of logics is to get some universal results that can be applied more or less directly to specific logics, this is one reason to call such a theory *universal logic*.

Some people may have the impression that such general universal results are trivial. This impression is generally due to the fact that these people have a concrete-oriented mind, and that something which is not specified has no meaning for them, and therefore universal logic appears as logical abstract nonsense. They are like someone who understands perfectly what is Felix, his cat, but for whom the concept of cat is a meaningless abstraction. This psychological limitation is in fact a strong defect because, as we have pointed through the example of the completeness theorem, what is trivial is generally the specific part, not the universal one which requires what is the fundamental capacity of human thought: abstraction.

7.2. Universal logic and universal algebra

Originally I introduced the terminology "universal logic" to denote a general theory of logics, by analogy with the expression "universal algebra" (cf. [1]).

What is universal algebra? During the XIXth century, lots of algebraic structures appeared and then some people started to turn this heterogeneous variety into a unified theory. In 1898, Whitehead wrote a book entitled *A treatise on universal algebra* (cf. [46]), but it is Garrett Birkhoff who is considered as the real founder of universal algebra [4]. Birkhoff was the first to give a very general definition of abstract algebra, as a set with a family of operators. He introduced further general concepts and proved several important universal results (see [14], [16]).

The idea beyond universal logic is to develop a general theory of logics in a similar way. This means that logics are considered as mathematical structures, general concepts are introduced and universal results are proved.

One central question is to know which kind of structures are logical structures. One may think that these structures are algebraic structures and that therefore universal logic is just a part of universal algebra, this was more or less the idea of Suszko. But as we have pointed out, it seems inappropriate to base essentially a general theory of logics on the notion of algebraic structures. Other types of structures are required.

7.3. Universal logic and the theory of structures

The idea I proposed about ten years ago is that logical structures must be considered as fundamental mother structures in the sense of Bourbaki, together with algebraic, topological and order structures. This was also the idea of a former student of de Possel, Jean Porte, 40 years ago (see [36]). In his work, Porte proposed several types of logical structures.

My idea was to focus on a logical structure of type $\langle \mathbb{S}; \vdash \rangle$ where $\vdash$ is a relation on $\mathcal{P}(\mathbb{S})X\mathbb{S}$. The important thing is that the structure of $\mathbb{S}$ is not specified, in fact, further on, any kind of structure can be put on $\mathbb{S}$, not only an algebraic structure. We are back therefore to something very close to Tarski's original theory of consequence operator. One important difference is that in this new definition of logical structure, no axioms are stated for the consequence relation $\vdash$, in the same way that no axioms are stated for the operators in Birkhoff's definition of abstract algebra.

Universal logic, like universal algebra, is just a part of the general theory of structures, logical abstract nonsense is a subfield of general abstract nonsense. If, as we have suggested, abstraction is the important thing, one could argue that what is really interesting is a general theory of structures, like category theory, and not a theory of specific structures like universal logic. The fact is that abstraction is really

[4]L. Corry erroneously says, in his otherwise excellent book [20], that the expression 'universal algebra" is due to Whitehead, this expression is due in fact to J.J. Sylvester, see [42].

a nonsense if it is considered only by itself. Abstraction is abstraction of something and when applied back it gives another view of this thing. Moreover there must be a continuous interplay between the specific and the general. Universal logic is an interesting material for the general theory of structures. For example, a central point in universal logic is to try to define properly a relation between logics which permits to compare them and to identify them (cf. [6], [13]). To solve this problem, new concepts and tools have to be introduced at the level of a general theory of structures, which can later on be applied to other fields of mathematics.

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