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Preface

This small book is a streamlined exposition of the notions and results leading to the construction of normal forms and, ultimately, to the construction of smooth conjugacies for the perturbations of tempered exponential dichotomies. These are exponential dichotomies for which the exponential growth rates of the underlying linear dynamics never vanish. In other words, its Lyapunov exponents are all nonzero. We consider mostly difference equations, although we also briefly consider the case of differential equations.

The main components of the exposition are tempered spectra, normal forms, and smooth conjugacies. The first two lie at the core of the theory and have an importance that undoubtedly surpasses the construction of conjugacies. Indeed, the theory is very rich and developed in various directions that are also of interest by themselves. This includes the study of dynamics with discrete and continuous time, of dynamics in finite and infinite-dimensional spaces, as well as of dynamics depending on a parameter. This led us to make an exposition not only of tempered spectra and subsequently of normal forms, but also briefly of some important developments in those other directions. Afterward, we continue the presentation with the construction of stable and unstable invariant manifolds and, consequently, of smooth conjugacies, while using most of the former material.

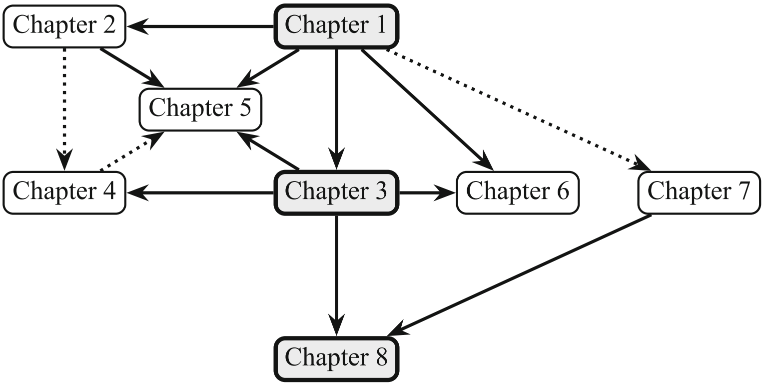
The text can be naturally divided into three parts. The first part (Chapters 1, 2, and 3, with emphasis on the basic theory) is dedicated to the tempered spectrum and the construction of normal forms. In Chapter 1, we introduce the notion of (tempered) spectrum in terms of the notion of tempered exponential dichotomy. The chapter also includes a description of all possible forms of the spectrum and detailed examples of all of them. We continue in Chapter 2 with the description of the Lyapunov exponents, which always belong to some connected component of the spectrum. We also consider exponentially decaying perturbations and show that again the Lyapunov exponents of the nonlinear dynamics belong to some connected component. Finally, in Chapter 3, starting with a block-diagonal preparation of the linear part, we construct normal forms for the tempered perturbations of a linear dynamics using an appropriate nonautonomous notion of resonance.

The second part (Chapters 4, 5, and 6, with emphasis on further developments) is dedicated to the discussion of some additional topics related to tempered spectra and normal forms. Although strictly speaking the material is not necessary for the third part, these developments are important by themselves and the presentation would be quite poorer without them. In Chapter 4, we consider dynamics depending on a parameter. In particular, we describe how the tempered spectrum may vary with a parameter-dependent linear perturbation, and we establish the regularity of the normal forms when the perturbation depends smoothly on the parameter. Chapter 5 is a brief presentation of the notions and results in the former chapters for differential equations. In Chapter 6, we consider the infinite-dimensional setting with the study of linear and nonlinear dynamics defined by sequences of compact linear operators and their perturbations. The study of perturbations depending on a parameter is of utmost importance and is the main theme for example of bifurcation theory. Normal forms play a crucial role in the study of bifurcations since they reduce the nonlinear part of the dynamics to the simplest possible form. Besides difference and differential equations, it is also important to consider infinite-dimensional systems both for discrete and continuous time. This includes partial differential equations and functional differential equations, although these topics clearly fall out of the scope of our book. For details we refer instead to the notes at the end of each chapter.

Finally, the third part (Chapters 7 and 8, with emphasis on smooth linearization) is dedicated to the construction of smooth conjugacies between a tempered exponential dichotomy and its tempered perturbations in the absence of resonances. This requires a detailed preparation in Chapter 7 with the construction of stable and unstable invariant manifolds together with crucial bounds. These are used in Chapter 8 to make a preparation of the dynamics so that the manifolds become the stable and unstable spaces. Finally, also in Chapter 8, we use the material in the former chapters on tempered spectra, formal forms, and invariant manifolds to construct smooth conjugacies when there are no resonances, or even when there are no resonances up to a given order.

We note that the notion of tempered spectrum is naturally adapted to the study of *nonautonomous* dynamics. The reason for this is that any autonomous linear dynamics with a tempered exponential dichotomy has automatically a uniform exponential dichotomy. We emphasize that in strong contrast to what happens with a uniform exponential dichotomy, for a tempered exponential dichotomy the stability along the stable direction when time goes forward and along the unstable direction when time goes backward need not be uniform. In other words, it may depend on the initial time. This causes important changes and the need for adaptations of the classical theory as well as for new ideas. Most notably, the spectra defined in terms of tempered exponential dichotomies and uniform exponential dichotomies are distinct in general. More precisely, the tempered spectrum may be smaller, which causes that it may lead to less resonances and thus to simpler normal forms (an explicit example is given in Chapter 3). Another important aspect is the need for Lyapunov norms in the study of exponentially decaying perturbations (see Chapter 2) and in the study of parameter-dependent dynamics (see Chapter 4). Other characteristics are the need for a spectral gap to obtain the regularity of the normal forms on a parameter in Chapter 4 and the need for a careful control of the small exponential terms in the construction of invariant manifolds in Chapter 7 and of smooth conjugacies in Chapter 8.

The following diagram is a summary of the relation between the chapters. A solid arrow means that there is a strong dependence between the two chapters, while a dotted arrow means that there is a dependence but small. The central line of the exposition with the discussion of tempered spectra, normal forms, and smooth conjugacies is marked in gray.



The text is self-contained, and all proofs have been simplified or even rewritten on purpose for the book so that all is as streamlined as possible. Moreover, all chapters are supplemented by detailed notes discussing the origins of the notions and results as well as their proofs, together with the discussion of the proper context, also with references to precursor results and further developments. The book is aimed at researchers and graduate students who wish to have a sufficiently broad view of the area, without the discussion of accessory material. It can also be used as a basis for graduate courses on spectra, normal forms, and smooth conjugacies.

Lisbon, Portugal Luís Barreira
November 2023 Claudia Valls

Contents

Chapter 1
Spectra and Examples

In this chapter we introduce the notions of tempered exponential dichotomy and of tempered spectrum for a sequence of $d \times d$ matrices that need not be invertible. The tempered spectrum can be thought of as a nonautonomous version of the usual notion of spectrum for a single matrix. We also describe all possible forms of the tempered spectrum and we give explicit examples of all of them. More precisely, for each possible form we describe explicitly a sequence of invertible matrices with that tempered spectrum.

1.1 Tempered Spectrum

We first introduce the notion of tempered exponential dichotomy. Let $(A_n)_{n\in\mathbb{Z}}$ be a sequence of $d \times d$ matrices (not necessarily invertible). For each $m, n \in \mathbb{Z}$ with $m \geq n$, we define

$$\mathcal{A}_{m,n} = \begin{cases} A_{m-1} \cdots A_n & \text{if } m > n, \\ \text{Id} & \text{if } m = n. \end{cases}$$

Definition 1.1 A sequence of $d \times d$ matrices $(A_n)_{n\in\mathbb{Z}}$ is said to have a *tempered exponential dichotomy* if:

1. There are projections $P_n \colon \mathbb{R}^d \to \mathbb{R}^d$ for $n \in \mathbb{Z}$ satisfying

$$A_n P_n = P_{n+1} A_n \quad \text{for each } n \in \mathbb{Z} \tag{1.1}$$

   such that the map

$$A_n|_{\ker P_n} \colon \ker P_n \to \ker P_{n+1} \tag{1.2}$$

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is onto and invertible.

2. There are a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that for each $m, n \in \mathbb{Z}$ we have

$$\|\mathcal{A}_{m,n} P_n\| \leq De^{-\lambda(m-n)+\varepsilon|n|} \quad \text{for } m \geq n \tag{1.3}$$

and

$$\|\bar{\mathcal{A}}_{m,n} Q_n\| \leq De^{-\lambda(n-m)+\varepsilon|n|} \quad \text{for } m \leq n, \tag{1.4}$$

where $Q_n = \text{Id} - P_n$ and

$$\bar{\mathcal{A}}_{m,n} = \left(\mathcal{A}_{n,m}|_{\ker P_m}\right)^{-1} : \ker P_n \to \ker P_m \quad \text{for } m \leq n. \tag{1.5}$$

Then we shall also say that $(A_n)_{n\in\mathbb{Z}}$ has a tempered exponential dichotomy with constants λ and D.

A sequence of positive numbers $(D_n)_{n\in\mathbb{Z}}$ is said to be *upper tempered* if

$$\limsup_{n\to\pm\infty} \frac{1}{|n|} \log D_n \leq 0.$$

Note that this happens if and only if given $\varepsilon > 0$, there is $D = D(\varepsilon) > 0$ such that

$$D_n \leq De^{\varepsilon|n|} \quad \text{for all } n \in \mathbb{Z}. \tag{1.6}$$

Thus, a sequence $(A_n)_{n\in\mathbb{Z}}$ has a tempered exponential dichotomy if and only if there are projections P_n for $n \in \mathbb{Z}$ satisfying (1.1) such that each map in (1.2) is onto and invertible, and there are $\lambda > 0$ and an upper tempered sequence $(D_n)_{n\in\mathbb{Z}}$ such that

$$\|\mathcal{A}_{m,n} P_n\| \leq D_n e^{-\lambda(m-n)} \quad \text{for } m \geq n$$

and

$$\|\bar{\mathcal{A}}_{m,n} Q_n\| \leq D_n e^{-\lambda(n-m)} \quad \text{for } m \leq n.$$

We shall also say that $(A_n)_{n\in\mathbb{Z}}$ has a *tempered exponential contraction* if it has a tempered exponential dichotomy with $P_n = \text{Id}$ for all $n \in \mathbb{Z}$ and that $(A_n)_{n\in\mathbb{Z}}$ has a *tempered exponential expansion* if it has a tempered exponential dichotomy with $P_n = 0$ for all $n \in \mathbb{Z}$. For any tempered exponential dichotomy, the sets

$$E_n = P_n(\mathbb{R}^d) \quad \text{and} \quad F_n = Q_n(\mathbb{R}^d)$$

are called, respectively, the *stable* and *unstable spaces* at time n. They satisfy

$$\mathbb{R}^d = E_n \oplus F_n \quad \text{for } n \in \mathbb{Z}$$

and can be univocally characterized as follows.

Proposition 1.1 *Assume that the sequence of $d \times d$ matrices $(A_n)_{n\in\mathbb{Z}}$ has a tempered exponential dichotomy. For each $n \in \mathbb{Z}$, we have*

$$E_n = \left\{ v \in \mathbb{R}^d : \sup_{m\geq n} \|\mathcal{A}_{m,n} v\| < +\infty \right\}$$

and F_n is the set of all $v \in \mathbb{R}^d$ for which there is a bounded sequence $(x_m)_{m\leq n}$ in $\mathbb{R}^d$ such that

$$x_n = v \quad \textit{and} \quad x_m = A_{m-1} x_{m-1} \textit{ for } m \leq n. \tag{1.7}$$

Proof Take $v \in E_n$. By (1.3) we have

$$\sup_{m\geq n} \|\mathcal{A}_{m,n} v\| < +\infty. \tag{1.8}$$

Now assume that $v \in \mathbb{R}^d$ satisfies (1.8). It follows from (1.3) that

$$\sup_{m\geq n} \|\mathcal{A}_{m,n} Q_n v\| = \sup_{m\geq n} \|\mathcal{A}_{m,n}(v - P_n v)\| < +\infty. \tag{1.9}$$

On the other hand, by (1.4), for $m \geq n$ we have

$$\|Q_n v\| \leq D e^{-\lambda(m-n)+\varepsilon|m|} \|\mathcal{A}_{m,n} Q_n v\|,$$

which is equivalent to

$$\|\mathcal{A}_{m,n} Q_n v\| \geq \frac{1}{D} e^{\lambda(m-n)-\varepsilon|m|} \|Q_n v\|.$$

If $Q_n v \neq 0$, then taking $\varepsilon < \lambda$ we obtain

$$\sup_{m\geq n} \|\mathcal{A}_{m,n} Q_n v\| = +\infty,$$

which contradicts (1.9). Hence, $Q_n v = 0$ and so $v \in E_n$.

Now we consider a vector $v \in F_n$ and the sequence $x_m = \bar{\mathcal{A}}_{m,n} v$ for $m \leq n$. Then property (1.7) holds and by (1.4) we have $\sup_{m\leq n} \|x_m\| < +\infty$. Finally, assume that $(x_m)_{m\leq n}$ is a sequence with the properties in the proposition. It follows from (1.1) and (1.3) that

$$\|P_n v\| = \|\mathcal{A}_{n,m} P_m x_m\| \leq D e^{-\lambda(n-m)+\varepsilon|m|} \|x_m\|$$

for $m \le n$. Taking $\varepsilon < \lambda$ and letting $\alpha = \sup_{m \le n} \|x_m\|$, we obtain

$$\|P_n v\| \le De^{-\lambda(n-m)+\varepsilon|m|}\alpha \to 0$$

when $m \to -\infty$. Hence, $P_n v = 0$ and so $v \in F_n$. □

The notion of tempered spectrum is defined in terms of the notion of tempered exponential dichotomy.

Definition 1.2 The *tempered spectrum* (or, simply, the *spectrum*) of a sequence of $d \times d$ matrices $A = (A_n)_{n \in \mathbb{Z}}$ is the set $\Sigma = \Sigma(A)$ of all numbers $a \in \mathbb{R}$ such that the sequence $(e^{-a} A_n)_{n \in \mathbb{Z}}$ does not have a tempered exponential dichotomy.

We note that the tempered spectrum of a constant sequence of matrices $A_n = B$ for $n \in \mathbb{Z}$ is the set of absolute values of the eigenvalues of B.

Given $a \in \mathbb{R}$ and $n \in \mathbb{Z}$, let

$$E_n^a = \left\{ v \in \mathbb{R}^d : \sup_{m \ge n} \left(e^{-a(m-n)} \|\mathcal{A}_{m,n} v\| \right) < +\infty \right\}$$

and let F_n^a be the set of all $v \in \mathbb{R}^d$ for which there is a sequence $(x_m)_{m \le n}$ in $\mathbb{R}^d$ satisfying (1.7) such that

$$\sup_{m \le n} (e^{-a(m-n)} \|x_m\|) < +\infty.$$

Clearly, if $a < b$, then

$$E_n^a \subset E_n^b \quad \text{and} \quad F_n^b \subset F_n^a \tag{1.10}$$

for $n \in \mathbb{Z}$. Now take $a \in \mathbb{R} \setminus \Sigma$. By Proposition 1.1,

$$\mathbb{R}^d = E_n^a \oplus F_n^a \quad \text{for } n \in \mathbb{Z} \tag{1.11}$$

is the splitting into stable and unstable spaces of the tempered exponential dichotomy of the sequence $(e^{-a} A_n)_{n \in \mathbb{Z}}$. Because of the invertibility of the maps in (1.2), the dimensions $\dim F_n^a$ (and so, by (1.11), also the dimensions $\dim E_n^a$) are independent of n. We shall denote their common value by $\dim F^a$.

Finally, the following result describes all possible forms of the tempered spectrum for an arbitrary sequence of matrices. For $-\infty \le a \le b \le +\infty$, let

$$|a, b| = \mathbb{R} \cap [a, b].$$

More precisely,